Appendix III

Useful Derived Theorems

III.1 Classical Propositional Logic

Postulating the axiomatization of *classical propositional logic* (CPL) discussed in Section 2.3, the following theorems over $\Delta \in \mathsf{obj}\ \mathbf{Sig}^{CPL}$ are provable:

(HS) $\{(p \to q), (q \to r)\} \vdash_{\Delta}^{CPL} p \to r$ (hypothetical syllogism)

Proof:

1.	$p \to q$	**Ass**
2.	$q \to r$	**Ass**
3.	$(q \to r) \to (p \to (q \to r))$	**A1-I**
4.	$p \to (q \to r)$	**R1-MP** 2, 3
5.	$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$	**A2-I**
6.	$(p \to q) \to (p \to r)$	**R1-MP** 4, 5
7.	$p \to r$	**R1-MP** 1, 6

(REFL) $\vdash_{\Delta}^{CPL} p \to p$ (reflexivity)

Proof:

1.	$p \to ((p \to p) \to p)$	**A1-I**
2.	$(p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p))$	**A2-I**
3.	$(p \to (p \to p)) \to (p \to p)$	**R1-MP** 1,2
4.	$p \to (p \to p)$	**A1-I**
5.	$p \to p$	**R1-MP** 4,3

(EXP) $\vdash_{\Delta}^{CPL} p \to ((p \to q) \to q)$ (expansion)

Proof:

1.	$p \to ((p \to q) \to p)$	**A1-I**
2.	$(p \to q) \to (p \to q)$	**REFL**
3.	$((p \to q) \to (p \to q)) \to (((p \to q) \to p) \to ((p \to q) \to q))$	**A2-I**
4.	$((p \to q) \to p) \to ((p \to q) \to q)$	**R1-MP** 2, 3
5.	$(((p \to q) \to p) \to ((p \to q) \to q)) \to$ $(p \to (((p \to q) \to p) \to ((p \to q) \to q)))$	**A1-I**
6.	$(p \to (((p \to q) \to p) \to ((p \to q) \to q))) \to$ $((p \to ((p \to q) \to p)) \to (p \to ((p \to q) \to q)))$	**A2-I**

7. $(((p \to q) \to p) \to ((p \to q) \to q)) \to$ $((p \to ((p \to q) \to p)) \to (p \to ((p \to q) \to q)))$ **HS** 5, 6
8. $(p \to ((p \to q) \to p)) \to (p \to ((p \to q) \to q))$ **R1-MP** 4, 7
9. $p \to ((p \to q) \to q)$ **R1-MP** 1, 8

(PERM) $\vdash_{\Delta}^{CPL}$ $(p \to (q \to r)) \to (q \to (p \to r))$ (permutation)

Proof:

1. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ **A2-I**
2. $((p \to q) \to (p \to r)) \to (q \to ((p \to q) \to (p \to r)))$ **A1-I**
3. $(p \to (q \to r)) \to (q \to ((p \to q) \to (p \to r)))$ **HS** 1, 2
4. $(q \to ((p \to q) \to (p \to r))) \to ((q \to (p \to q)) \to (q \to (p \to r)))$ **A2-I**
5. $q \to (p \to q)$ **A1-I**
6. $(q \to (p \to q)) \to$ $(((q \to (p \to q)) \to (q \to (p \to r))) \to (q \to (p \to r)))$ **EXP**
7. $((q \to (p \to q)) \to (q \to (p \to r))) \to (q \to (p \to r))$ **R1-MP** 5, 6
8. $(q \to ((p \to q) \to (p \to r))) \to (q \to (p \to r))$ **HS** 4, 7
9. $(p \to (q \to r)) \to (q \to (p \to r))$ **HS** 3, 8

(LTRAN) $\vdash_{\Delta}^{CPL}$ $(p \to q) \to ((r \to p) \to (r \to q))$ (left transitivity)

Proof:

1. $(p \to q) \to (r \to (p \to q))$ **A1-I**
2. $(r \to (p \to q)) \to ((r \to p) \to (r \to q))$ **A2-I**
3. $(p \to q) \to ((r \to p) \to (r \to q))$ **HS** 1, 2

(RTRAN) $\vdash_{\Delta}^{CPL}$ $(p \to q) \to ((q \to r) \to (p \to r))$ (right transitivity)

Proof:

1. $(q \to r) \to ((p \to q) \to (p \to r))$ **LTRAN**
2. $((q \to r) \to ((p \to q) \to (p \to r))) \to$ $((p \to q) \to ((q \to r) \to (p \to r)))$ **PERM**
3. $(p \to q) \to ((q \to r) \to (p \to r))$ **R1-MP** 1, 2

(CONT) $\vdash_{\Delta}^{CPL}$ $(p \to (p \to q)) \to (p \to q)$ (contraction)

Proof:

1. $p \to ((p \to q) \to q)$ **EXP**
2. $(p \to ((p \to q) \to q)) \to ((p \to (p \to q)) \to (p \to q))$ **A2-I**
3. $(p \to (p \to q)) \to (p \to q)$ **R1-MP** 1, 2

(NEG-L) $\vdash_{\Delta}^{CPL}$ $p \to (\neg p \to q)$

Proof:

1. $\neg p \to (\neg q \to \neg p)$ **A1-I**
2. $(\neg q \to \neg p) \to (p \to q)$ **A3-N**
3. $((\neg q \to \neg p) \to (p \to q)) \to$ $((\neg p \to (\neg q \to \neg p)) \to (\neg p \to (p \to q)))$ **LTRAN**
4. $(\neg p \to (\neg q \to \neg p)) \to (\neg p \to (p \to q))$ **R1-MP** 2, 3
5. $\neg p \to (p \to q))$ **R1-MP** 1, 4

6. $(\neg p \to (p \to q)) \to (p \to (\neg p \to q))$ **PERM**
7. $p \to (\neg p \to q)$ **R1-MP** 5, 6

(DOUB) $\vdash^{CPL}_{\Delta} \neg\neg p \to p$ (double negation)

Proof:

1. $\neg\neg p \to (\neg\neg\neg\neg p \to \neg\neg p)$ **A1-I**
2. $(\neg\neg\neg\neg p \to \neg\neg p) \to (\neg p \to \neg\neg\neg p)$ **A3-N**
3. $\neg\neg p \to (\neg p \to \neg\neg\neg p)$ **HS** 1, 2
4. $(\neg p \to \neg\neg\neg p) \to (\neg\neg p \to p)$ **A3-N**
5. $\neg\neg p \to (\neg\neg p \to p)$ **HS** 3, 4
6. $(\neg\neg p \to (\neg\neg p \to p)) \to (\neg\neg p \to p)$ **CONT**
7. $\neg\neg p \to p$ **R1-MP** 5, 6

(NEG-R) $\vdash^{CPL}_{\Delta} (p \to q) \to ((p \to \neg q) \to \neg p)$

Proof:

1. $q \to (\neg q \to \neg p)$ **NEG-L**
2. $(q \to (\neg q \to \neg p)) \to ((p \to q) \to (p \to (\neg q \to \neg p)))$ **LTRAN**
3. $(p \to q) \to (p \to (\neg q \to \neg p))$ **R1-MP** 1, 2
4. $(p \to (\neg q \to \neg p)) \to ((p \to \neg q) \to (p \to \neg p))$ **A2-I**
5. $(p \to q) \to ((p \to \neg q) \to (p \to \neg p))$ **HS** 3, 4
6. $p \to (\neg p \to \neg(p \to \neg p))$ **NEG-L**
7. $(p \to (\neg p \to \neg(p \to \neg p))) \to ((p \to \neg p) \to (p \to \neg(p \to \neg p)))$ **A2-I**
8. $(p \to \neg p) \to (p \to \neg(p \to \neg p))$ **R1-MP** 6, 7
9. $\neg\neg p \to p$ **DOUB**
10. $(\neg\neg p \to p) \to ((p \to \neg(p \to \neg p)) \to (\neg\neg p \to \neg(p \to \neg p)))$ **A2-I**
11. $(p \to \neg(p \to \neg p)) \to (\neg\neg p \to \neg(p \to \neg p))$ **R1-MP** 9, 10
12. $(p \to \neg p) \to (\neg\neg p \to \neg(p \to \neg p))$ **HS** 8, 11
13. $(\neg\neg p \to \neg(p \to \neg p)) \to ((p \to \neg p) \to \neg p)$ **A3-N**
14. $(p \to \neg p) \to ((p \to \neg p) \to \neg p)$ **HS** 12, 13
15. $((p \to \neg p) \to ((p \to \neg p) \to \neg p)) \to ((p \to \neg p) \to \neg p)$ **CONT**
16. $(p \to \neg p) \to \neg p$ **R1-MP** 14, 15
17. $((p \to \neg p) \to \neg p) \to$ **LTRAN**
    $(((p \to \neg q) \to (p \to \neg p)) \to ((p \to \neg q) \to \neg p))$
18. $((p \to \neg q) \to (p \to \neg p)) \to ((p \to \neg q) \to \neg p)$ **R1-MP** 16, 17
19. $(p \to q) \to ((p \to \neg q) \to \neg p)$ **HS** 5, 18

(CONP) $\vdash^{CPL}_{\Delta} (p \to q) \to (\neg q \to \neg p)$ (contrapositive)

Proof:

1. $\neg q \to (p \to \neg q)$ **A1-I**
2. $(p \to q) \to ((p \to \neg q) \to \neg p)$ **NEG-R**
3. $((p \to \neg q) \to \neg p) \to ((\neg q \to (p \to \neg q)) \to (\neg q \to \neg p))$ **LTRAN**
4. $(((p \to \neg q) \to \neg p) \to ((\neg q \to (p \to \neg q)) \to (\neg q \to \neg p))) \to$ **PERM**
   $((\neg q \to (p \to \neg q)) \to (((p \to \neg q) \to \neg p) \to (\neg q \to \neg p)))$
5. $(\neg q \to (p \to \neg q)) \to (((p \to \neg q) \to \neg p) \to (\neg q \to \neg p))$ **R1-MP** 3, 4
6. $((p \to \neg q) \to \neg p) \to (\neg q \to \neg p)$ **R1-MP** 1, 5
7. $(p \to q) \to (\neg q \to \neg p)$ **HS** 2, 6

(OR-L) $\{p \to q, r \to q\} \vdash_{\Delta}^{CPL} p \vee r \to q$

Proof:

1. $p \to q$ **Ass**
2. $r \to q$ **Ass**
3. $(r \to q) \to (\neg p \to (r \to q))$ **A1-I**
4. $\neg p \to (r \to q)$ **R1-MP** 2, 3
5. $(\neg p \to (r \to q)) \to ((\neg p \to r) \to (\neg p \to q))$ **A2-I**
6. $(\neg p \to r) \to (\neg p \to q)$ **R1-MP** 4, 5
7. $(p \to q) \to (\neg q \to \neg p)$ **CONP**
8. $\neg q \to \neg p$ **R1-MP** 1, 7
9. $(\neg q \to \neg p) \to ((\neg q \to \neg\neg p) \to \neg\neg q)$ **NEG-R**
10. $(\neg q \to \neg\neg p) \to \neg\neg q$ **R1-MP** 8, 9
11. $\neg\neg q \to q$ **DOUB**
12. $(\neg q \to \neg\neg p) \to q$ **HS** 10, 11
13. $(\neg p \to q) \to (\neg q \to \neg\neg p)$ **CONP**
14. $(\neg p \to q) \to q$ **HS** 13, 12
15. $(\neg p \to r) \to q$ **HS** 6, 14
16. $p \vee r \to q$ **D3-OR** 15

(OR-R) $\{p \to q\} \vdash_{\Delta}^{CPL} p \to q \vee r$ [or $\{p \to q\} \vdash_{\Delta}^{CPL} p \to r \vee q$]

Proof:

1. $p \to q$ **Ass**
2. $q \to (\neg q \to r)$ **NEG-L**
3. $p \to (\neg q \to r)$ **HS** 1, 2
4. $p \to q \vee r$ **D3-OR** 3

(AND-L) $\{p \to q\} \vdash_{\Delta}^{CPL} p \wedge r \to q$ [or $\{p \to q\} \vdash_{\Delta}^{CPL} r \wedge p \to q$]

Proof:

1. $p \to q$ **Ass**
2. $(p \to q) \to (\neg q \to \neg p)$ **CONP**
3. $\neg q \to \neg p$ **R1-MP** 1, 2
4. $p \to (\neg p \to \neg r)$ **NEG-L**
5. $(p \to (\neg p \to \neg r)) \to (\neg p \to (p \to \neg r))$ **PERM**
6. $\neg p \to (p \to \neg r)$ **R1-MP** 4, 5
7. $\neg q \to (p \to \neg r)$ **HS** 3, 6
8. $\neg\neg\neg(p \to \neg r) \to \neg(p \to \neg r)$ **DOUB**
9. $(\neg\neg\neg(p \to \neg r) \to \neg(p \to \neg r)) \to ((p \to \neg r) \to \neg\neg(p \to \neg r))$ **A3-N**
10. $(p \to \neg r) \to \neg\neg(p \to \neg r)$ **R1-MP** 8, 9
11. $\neg q \to \neg\neg(p \to \neg r)$ **HS** 7, 10
12. $(\neg q \to \neg\neg(p \to \neg r)) \to (\neg(p \to \neg r) \to q)$ **A3-N**
13. $\neg(p \to \neg r) \to q$ **R1-MP** 11, 12
14. $p \wedge r \to q$ **D4-AND** 13

(AND-R) $\{p \to q, p \to r\} \vdash_{\triangle}^{CPL} p \to q \wedge r$

Proof:

1.	$p \to q$	**Ass**
2.	$p \to r$	**Ass**
3.	$(p \to q) \to ((q \to \neg r) \to (p \to \neg r))$	**RTRAN**
4.	$(q \to \neg r) \to (p \to \neg r)$	**R1-MP** 1, 3
5.	$(p \to \neg r) \to ((p \to \neg\neg r) \to \neg p)$	**NEG-R**
6.	$((p \to \neg r) \to ((p \to \neg\neg r) \to \neg p)) \to$ $((p \to \neg\neg r) \to ((p \to \neg r) \to \neg p))$	**PERM**
7.	$(p \to \neg\neg r) \to ((p \to \neg r) \to \neg p)$	**R1-MP** 5, 6
8.	$\neg\neg\neg r \to \neg r$	**DOUB**
9.	$(\neg\neg\neg r \to \neg r) \to (r \to \neg\neg r)$	**A3-N**
10.	$r \to \neg\neg r$	**R1-MP** 8, 9
11.	$p \to \neg\neg r$	**HS** 2, 10
12.	$(p \to \neg r) \to \neg p$	**R1-MP** 11, 7
13.	$(q \to \neg r) \to \neg p$	**HS** 4, 12
14.	$((q \to \neg r) \to \neg p) \to (\neg\neg p \to \neg(q \to \neg r))$	**CONP**
15.	$\neg\neg p \to \neg(q \to \neg r)$	**R1-MP** 13, 14
16.	$\neg\neg\neg p \to \neg p$	**DOUB**
17.	$(\neg\neg\neg p \to \neg p) \to (p \to \neg\neg p)$	**A3-N**
18.	$p \to \neg\neg p$	**R1-MP** 16, 17
19.	$p \to \neg(q \to \neg r)$	**HS** 18, 15
20.	$p \to q \wedge r$	**D4-AND** 19

(AND-E) $\{p \wedge q\} \vdash_{\triangle}^{CPL} p$ [or $\{p \wedge q\} \vdash_{\triangle}^{CPL} q$]

Proof:

1.	$p \wedge q$	**Ass**
2.	$\neg(p \to \neg q)$	**D4-AND** 1
3.	$\neg(p \to \neg q) \to (\neg p \to \neg(p \to \neg q))$	**A1-I**
4.	$\neg p \to \neg(p \to \neg q)$	**R1-MP** 2, 3
5.	$(\neg p \to \neg(p \to \neg q)) \to ((p \to \neg q) \to p)$	**A3-N**
6.	$(p \to \neg q) \to p$	**R1-MP** 4, 5
7.	$p \to (\neg p \to \neg q)$	**NEG-L**
8.	$(p \to (\neg p \to \neg q)) \to (\neg p \to (p \to \neg q))$	**PERM**
9.	$\neg p \to (p \to \neg q)$	**R1-MP** 7, 8
10.	$\neg p \to p$	**HS** 9, 6
11.	$(\neg p \to p) \to ((\neg p \to \neg p) \to \neg\neg p)$	**NEG-R**
12.	$(\neg p \to \neg p) \to \neg\neg p$	**R1-MP** 10, 11
13.	$\neg p \to \neg p$	**REFL**
14.	$\neg\neg p$	**R1-MP** 13, 12
15.	$\neg\neg p \to p$	**DOUB**
16.	p	**R1-MP** 14, 15

(AND-I) $\{p, q\} \vdash_{\triangle}^{CPL} p \wedge q$

Proof:

1.	p	**Ass**
2.	q	**Ass**

3. $p \to ((r \to r) \to p)$ **A1-I**
4. $(r \to r) \to p$ **R1-MP** 1, 3
5. $q \to ((r \to q) \to q)$ **A1-I**
6. $(r \to r) \to q$ **R1-MP** 2, 5
7. $(r \to r) \to p \land q$ **AND-R** 4, 6
8. $r \to r$ **REFL**
9. $p \land q$ **R1-MP** 8, 7

(IFF-RL) $\{p \to q, q \to p\} \vdash_{\triangle}^{CPL} p \leftrightarrow q$

Proof:

1. $p \to q$ **Ass**
2. $q \to p$ **Ass**
3. $(p \to q) \land (q \to p)$ **AND-I** 1, 2
4. $p \leftrightarrow q$ **D5-IFF** 3

(IFF-E) $\{p \leftrightarrow q\} \vdash_{\triangle}^{CPL} p \to q$ [or $\{p \leftrightarrow q\} \vdash_{\triangle}^{CPL} q \to p$]

Proof:

1. $p \leftrightarrow q$ **Ass**
2. $(p \to q) \land (q \to p)$ **D5-IFF** 1
3. $p \to q$ **AND-E** 2

(DM) $\vdash_{\triangle}^{CPL} \neg(p \lor q) \leftrightarrow \neg p \land \neg q$ [or $\neg(p \land q) \leftrightarrow \neg p \lor \neg q$] (De Morgan)

Proof:

1. $p \to p$ **REFL**
2. $p \to p \lor q$ **OR-R** 1
3. $(p \to p \lor q) \to (\neg(p \lor q) \to \neg p)$ **CONP**
4. $\neg(p \lor q) \to \neg p$ **R1-MP** 2, 3
5. $q \to q$ **REFL**
6. $q \to p \lor q$ **OR-R** 5
7. $(q \to p \lor q) \to (\neg(p \lor q) \to \neg q)$ **CONP**
8. $\neg(p \lor q) \to \neg q$ **R1-MP** 6, 7
9. $\neg(p \lor q) \to \neg p \land \neg q$ **AND-R** 4, 8
10. $\neg\neg\neg q \to \neg q$ **DOUB**
11. $(\neg\neg\neg q \to \neg q) \to (q \to \neg\neg q)$ **A3-N**
12. $q \to \neg\neg q$ **R1-MP** 10, 11
13. $(q \to \neg\neg q) \to ((\neg p \to q) \to (\neg p \to \neg\neg q))$ **LTRAN**
14. $(\neg p \to q) \to (\neg p \to \neg\neg q)$ **R1-MP** 12, 13
15. $((\neg p \to q) \to (\neg p \to \neg\neg q)) \to$ $(\neg(\neg p \to \neg\neg q) \to \neg(\neg p \to q))$ **CONP**
16. $\neg(\neg p \to \neg\neg q) \to \neg(\neg p \to q)$ **R1-MP** 13, 15
17. $\neg p \land \neg q \to \neg(\neg p \to q)$ **D4-AND** 16
18. $p \to (\neg p \to q)$ **NEG-L**
19. $q \to (\neg p \to q)$ **A1-I**
20. $p \lor q \to (\neg p \to q)$ **OR-L** 18, 19
21. $(p \lor q \to (\neg p \to q)) \to (\neg(\neg p \to q) \to \neg(p \lor q))$ **CONP**
22. $\neg(\neg p \to q) \to \neg(p \lor q)$ **R1-MP** 20, 21

23. $\neg p \wedge \neg q \to \neg(p \vee q)$ **HS** 17, 22
24. $\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$ **IFF-RL** 9, 23

(DIST-OA) $\vdash_{\Delta}^{CPL}$ $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$[1]
(DIST-AO) [or $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$]
(distribution of $\vee$ over $\wedge$)

Proof:

1. $p \to p$ **REFL**
2. $p \to p \vee q$ **OR-R** 1
3. $q \to q$ **REFL**
4. $q \wedge r \to q$ **AND-L** 3
5. $q \wedge r \to p \vee q$ **OR-R** 4
6. $p \vee (q \wedge r) \to (p \vee q) \wedge (p \vee r)$ **OR-L** 2, 5
7. $q \to ((q \to \neg r) \to \neg r)$ **EXP**
8. $((q \to \neg r) \to \neg r) \to (\neg\neg r \to \neg(q \to \neg r))$ **CONP**
9. $q \to (\neg\neg r \to \neg(q \to \neg r))$ **HS** 7, 8
10. $\neg\neg\neg r \to \neg r$ **DOUB**
11. $(\neg\neg\neg r \to \neg r) \to (r \to \neg\neg r)$ **A3-N**
12. $r \to \neg\neg r$ **R1-MP** 10, 11
13. $(r \to \neg\neg r) \to$ $((\neg\neg r \to \neg(q \to \neg r)) \to (r \to \neg(q \to \neg r)))$ **RTRAN**
14. $(\neg\neg r \to \neg(q \to \neg r)) \to (r \to \neg(q \to \neg r))$ **R1-MP** 12, 13
15. $q \to (r \to \neg(q \to \neg r))$ **HS** 9, 14
16. $(q \to (r \to \neg(q \to \neg r))) \to$ $((\neg p \to q) \to (\neg p \to (r \to \neg(q \to \neg r))))$ **LTRAN**
17. $(\neg p \to q) \to (\neg p \to (r \to \neg(q \to \neg r)))$ **R1-MP** 15, 16
18. $(\neg p \to (r \to \neg(q \to \neg r))) \to$ $((\neg p \to r) \to (\neg p \to \neg(q \to \neg r)))$ **A2-I**
19. $(\neg p \to q) \to ((\neg p \to r) \to (\neg p \to \neg(q \to \neg r)))$ **HS** 17, 18
20. $((\neg p \to r) \to (\neg p \to \neg(q \to \neg r))) \to$ $(\neg(\neg p \to \neg(q \to \neg r)) \to \neg(\neg p \to r))$ **CONP**
21. $(\neg p \to q) \to (\neg(\neg p \to \neg(q \to \neg r)) \to \neg(\neg p \to r))$ **HS** 19, 20
22. $((\neg p \to q) \to (\neg(\neg p \to \neg(q \to \neg r)) \to \neg(\neg p \to r))) \to$ $(\neg(\neg p \to \neg(q \to \neg r)) \to ((\neg p \to q) \to \neg(\neg p \to r)))$ **PERM**
23. $\neg(\neg p \to \neg(q \to \neg r)) \to ((\neg p \to q) \to \neg(\neg p \to r))$ **R1-MP** 21, 22
24. $(\neg(\neg p \to \neg(q \to \neg r)) \to ((\neg p \to q) \to \neg(\neg p \to r))) \to$ $\neg((\neg p \to q) \to \neg(\neg p \to r)) \to \neg\neg(\neg p \to \neg(q \to \neg r))$ **CONP**
25. $\neg((\neg p \to q) \to \neg(\neg p \to r)) \to \neg\neg(\neg p \to \neg(q \to \neg r))$ **R1-MP** 23, 24
26. $\neg\neg(\neg p \to \neg(q \to \neg r)) \to (\neg p \to \neg(q \to \neg r))$ **DOUB**
27. $\neg((\neg p \to q) \to \neg(\neg p \to r)) \to (\neg p \to \neg(q \to \neg r))$ **HS** 25, 26
28. $(p \vee q) \wedge (p \vee r) \to p \vee (q \wedge r)$ **D3-OR**, **D4-AND** 27
29. $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$ **IFF-RL** 6, 28

[1] A sentence with p at the right-hand side of each sub-formula is also provable.

(DIST-IFA) $\vdash^{CPL}_{\Delta}$ $(p \to (q \wedge r)) \leftrightarrow (p \to q) \wedge (p \to r)$
(DIST-IFO) [or $(p \to (q \vee r)) \leftrightarrow (p \to q) \vee (p \to r)$]
(distribution of implication over $\wedge$ and $\vee$)

Proof:

1. $p \wedge (\neg q \vee \neg r) \to (p \wedge \neg q) \vee (p \wedge \neg r)$ **DIST-AO**
2. $\neg(p \to \neg(\neg q \vee \neg r)) \to \neg(p \to \neg\neg q) \vee \neg(p \to \neg\neg r)$ **D4-AND** 1
3. $\neg(p \to \neg\neg q) \vee \neg(p \to \neg\neg r) \leftrightarrow \neg((p \to \neg\neg q) \wedge (p \to \neg\neg r))$ **DM**
4. $\neg(p \to \neg\neg q) \vee \neg(p \to \neg\neg r) \to \neg((p \to \neg\neg q) \wedge (p \to \neg\neg r))$ **IFF-E** 3
5. $\neg(p \to \neg(\neg q \vee \neg r)) \to \neg((p \to \neg\neg q) \wedge (p \to \neg\neg r))$ **HS** 2, 4
6. $(\neg(p \to \neg(\neg q \vee \neg r)) \to \neg((p \to \neg\neg q) \wedge (p \to \neg\neg r))) \to$ **A3-N**
   $((p \to \neg\neg q) \wedge (p \to \neg\neg r) \to (p \to \neg(\neg q \vee \neg r)))$
7. $(p \to \neg\neg q) \wedge (p \to \neg\neg r) \to (p \to \neg(\neg q \vee \neg r))$ **R1-MP** 5, 6
8. $\neg(q \wedge r) \leftrightarrow (\neg q \vee \neg r)$ **DM**
9. $\neg(q \wedge r) \to (\neg q \vee \neg r)$ **IFF-E** 8
10. $\neg\neg\neg(\neg q \vee \neg r) \to \neg(\neg q \vee \neg r)$ **DOUB**
11. $(\neg\neg\neg(\neg q \vee \neg r) \to \neg(\neg q \vee \neg r)) \to ((\neg q \vee \neg r) \to \neg\neg(\neg q \vee \neg r))$ **A3-N**
12. $(\neg q \vee \neg r) \to \neg\neg(\neg q \vee \neg r)$ **R1-MP** 10, 11
13. $\neg(q \wedge r) \to \neg\neg(\neg q \vee \neg r)$ **HS** 9, 12
14. $(\neg(q \wedge r) \to \neg\neg(\neg q \vee \neg r)) \to (\neg(\neg q \vee \neg r) \to q \wedge r)$ **A3-N**
15. $\neg(\neg q \vee \neg r) \to q \wedge r$ **R1-MP** 13, 14
16. $(\neg(\neg q \vee \neg r) \to q \wedge r) \to ((p \to \neg(\neg q \vee \neg r)) \to (p \to q \wedge r))$ **LTRAN**
17. $(p \to \neg(\neg q \vee \neg r)) \to (p \to q \wedge r)$ **R1-MP** 15, 16
18. $(p \to \neg\neg q) \wedge (p \to \neg\neg r) \to (p \to q \wedge r)$ **HS** 7, 17
19. $\neg\neg\neg q \to \neg q$ **DOUB**
20. $(\neg\neg\neg q \to \neg q) \to (q \to \neg\neg q)$ **A3-N**
21. $q \to \neg\neg q$ **R1-MP** 19, 20
22. $(q \to \neg\neg q) \to ((p \to q) \to (p \to \neg\neg q))$ **LTRAN**
23. $(p \to q) \to (p \to \neg\neg q)$ **R1-MP** 21, 22
24. $(p \to q) \wedge (p \to r) \to (p \to \neg\neg q)$ **AND-L** 23
25. $\neg\neg\neg r \to \neg r$ **DOUB**
26. $(\neg\neg\neg r \to \neg r) \to (r \to \neg\neg r)$ **A3-N**
27. $r \to \neg\neg r$ **R1-MP** 25, 26
28. $(r \to \neg\neg r) \to ((p \to r) \to (p \to \neg\neg r))$ **LTRAN**
29. $(p \to r) \to (p \to \neg\neg r)$ **R1-MP** 27, 28
30. $(p \to q) \wedge (p \to r) \to (p \to \neg\neg r)$ **AND-L** 29
31. $(p \to q) \wedge (p \to r) \to (p \to \neg\neg q) \wedge (p \to \neg\neg r)$ **AND-R** 24, 30
32. $(p \to q) \wedge (p \to r) \to (p \to q \wedge r)$ **HS** 18, 31
33. $q \to q$ **REFL**
34. $q \wedge r \to q$ **AND-L** 33
35. $(q \wedge r \to q) \to ((p \to q \wedge r) \to (p \to q))$ **LTRAN**
36. $(p \to q \wedge r) \to (p \to q)$ **R1-MP** 34, 35
37. $r \to r$ **REFL**
38. $q \wedge r \to r$ **AND-L** 37
39. $(q \wedge r \to r) \to ((p \to q \wedge r) \to (p \to r))$ **LTRAN**
40. $(p \to q \wedge r) \to (p \to r)$ **R1-MP** 38, 39
41. $(p \to q \wedge r) \to (p \to q) \wedge (p \to r)$ **AND-R** 36, 40
42. $(p \to q \wedge r) \leftrightarrow (p \to r) \wedge (p \to r)$ **IFF-RL** 32, 42

(REPL-CPL) $\{x \leftrightarrow y\} \vdash_{\Delta}^{CPL} p[q\backslash x] \leftrightarrow p[q\backslash y]$ (replacement)

We prove this theorem by structural induction on $\mathcal{G}^{CPL}(\Delta)$. The base case is determined by those propositions p such that $p \in \mathcal{E}^{CPL}(\Delta)$, meaning that p is a proposition symbol. We have to examine two circumstances:

i) If $p \equiv q$ then:

1.	$x \leftrightarrow y$	**Ass**
2.	$(x \leftrightarrow y) \rightarrow (x \leftrightarrow y)$	**REFL**
3.	$x \leftrightarrow y$	**R1-MP** 1, 2

ii) If $p \not\equiv q$ then:

4.	$x \leftrightarrow y$	**Ass**
5.	$p \rightarrow p$	**REFL**
6.	$p \leftrightarrow p$	**IFF-RL** 5, 5
7.	$(p \leftrightarrow p) \rightarrow ((x \leftrightarrow y) \rightarrow (p \leftrightarrow p))$	**A1-I**
8.	$(x \leftrightarrow y) \rightarrow (p \leftrightarrow p)$	**R1-MP** 6, 7
9.	$p \leftrightarrow p$	**R1-MP** 4, 8

Therefore, $\{x \leftrightarrow y\} \vdash_{\Delta}^{CPL} p[q\backslash x] \leftrightarrow p[q\backslash y]$ for any $p \in \mathcal{E}^{CPL}(\Delta)$ and $q \in \mathcal{G}^{CPL}(\Delta)$. Now, the inductive step is developed as follows for $p \equiv \neg r$:

1.	$x \leftrightarrow y$	**Ass**
2.	$r[q\backslash x] \leftrightarrow r[q\backslash y]$	**Ind. Hyp.** 1
3.	$r[q\backslash x] \rightarrow r[q\backslash y]$	**IFF-E** 2
4.	$(r[q\backslash x] \rightarrow r[q\backslash y]) \rightarrow (\neg r[q\backslash y] \rightarrow \neg r[q\backslash x])$	**CONP**
5.	$\neg r[q\backslash y] \rightarrow \neg r[q\backslash x]$	**R1-MP** 3, 4
6.	$r[q\backslash y] \rightarrow r[q\backslash x]$	**IFF-E** 2
7.	$(r[q\backslash y] \rightarrow r[q\backslash x]) \rightarrow (\neg r[q\backslash x] \rightarrow \neg r[q\backslash y])$	**CONP**
8.	$\neg r[q\backslash x] \rightarrow \neg r[q\backslash y]$	**R1-MP** 6, 7
9.	$\neg r[q\backslash x] \leftrightarrow \neg r[q\backslash y]$	**IFF-RL** 6, 9
10.	$(\neg r)[q\backslash x] \leftrightarrow (\neg r)[q\backslash y]$	**DEF** $[\cdot]$ 9

If $p \equiv r \rightarrow s$ then:

1.	$x \leftrightarrow y$	**Ass**
2.	$r[q\backslash x] \leftrightarrow r[q\backslash y]$	**Ind. Hyp.** 1
3.	$s[q\backslash x] \leftrightarrow s[q\backslash y]$	**Ind. Hyp.** 1
4.	$r[q\backslash y] \rightarrow r[q\backslash x]$	**IFF-E** 2
5.	$s[q\backslash x] \rightarrow s[q\backslash y]$	**IFF-E** 3
6.	$(r[q\backslash y] \rightarrow r[q\backslash x]) \rightarrow ((r[q\backslash x] \rightarrow s[q\backslash x]) \rightarrow (r[q\backslash y] \rightarrow s[q\backslash x]))$	**RTRAN**
7.	$(r[q\backslash x] \rightarrow s[q\backslash x]) \rightarrow (r[q\backslash y] \rightarrow s[q\backslash x])$	**R1-MP** 5, 6
8.	$(s[q\backslash x] \rightarrow s[q\backslash y]) \rightarrow ((r[q\backslash y] \rightarrow s[q\backslash x]) \rightarrow (r[q\backslash y] \rightarrow s[q\backslash y]))$	**RTRAN**
9.	$(r[q\backslash y] \rightarrow s[q\backslash x]) \rightarrow (r[q\backslash y] \rightarrow s[q\backslash y])$	**R1-MP** 5, 8
10.	$(r[q\backslash x] \rightarrow s[q\backslash x]) \rightarrow (r[q\backslash y] \rightarrow s[q\backslash y])$	**HS** 7, 9
11.	$r[q\backslash x] \rightarrow r[q\backslash y]$	**IFF-E** 2
12.	$s[q\backslash y] \rightarrow s[q\backslash x]$	**IFF-E** 3
13.	$(s[q\backslash y] \rightarrow s[q\backslash x]) \rightarrow ((r[q\backslash y] \rightarrow s[q\backslash y]) \rightarrow (r[q\backslash y] \rightarrow s[q\backslash x]))$	**RTRAN**
14.	$(r[q\backslash y] \rightarrow s[q\backslash y]) \rightarrow (r[q\backslash y] \rightarrow s[q\backslash x])$	**R1-MP** 12, 13

15. $(r[q\backslash x] \to r[q\backslash y]) \to$ **RTRAN**
    $((r[q\backslash y] \to s[q\backslash x]) \to (r[q\backslash x] \to s[q\backslash x]))$
16. $(r[q\backslash y] \to s[q\backslash x]) \to (r[q\backslash x] \to s[q\backslash x])$ **R1-MP** 12, 15
17. $(r[q\backslash y] \to s[q\backslash y]) \to (r[q\backslash x] \to s[q\backslash x])$ **HS** 14, 16
18. $(r[q\backslash x] \to s[q\backslash x]) \leftrightarrow (r[q\backslash y] \to s[q\backslash y])$ **IFF-RL** 10, 17
19. $(r \to s)[q\backslash x] \leftrightarrow (r \to s)[q\backslash y]$ **DEF** $[\cdot]$ 18

Since the other connectives are all defined as abbreviations, they do not need to be treated. Therefore, we can conclude that $\{x \leftrightarrow y\} \vdash^{CPL}_{\Delta} p[q\backslash x] \leftrightarrow p[q\backslash y]$ for any $\{p, q, x, y\} \subseteq \mathcal{G}^{CPL}(\Delta)$.

III.2 Propositional Linear Time Logic

Postulating the axiomatization of *linear time propositional logic* ($PLTL$) discussed in Section 2.4, the following theorems over $\Delta \in \mathsf{obj}\ \mathbf{Sig}^{PLTL}$ are provable:

(REPL-PLTL) $\{x \leftrightarrow y\} \vdash^{PLTL}_{\Delta} p[q\backslash x] \leftrightarrow p[q\backslash y]$ (replacement)

We prove this theorem by structural induction on $\mathcal{G}^{PLTL}(\Delta)$. The proof is essentially the same as for CPL, with the base case extended to consider **beg** and with an additional inductive step. We have to examine here the case where $p \equiv r\mathbf{V}s$:

1. $x \leftrightarrow y$ **Ass**
2. $r[q\backslash x] \leftrightarrow r[q\backslash y]$ **Ind. Hyp.** 1
3. $s[q\backslash x] \leftrightarrow s[q\backslash y]$ **Ind. Hyp.** 1
4. $r[q\backslash x] \to r[q\backslash y]$ **AND-E** 2
5. $s[q\backslash x] \to s[q\backslash y]$ **AND-E** 3
6. $\mathbf{G}(r[q\backslash x] \to r[q\backslash y])$ **R2-G** 4
7. $\mathbf{G}(r[q\backslash x] \to r[q\backslash y]) \to$ **A4-GV**
   $((r[q\backslash x])\mathbf{V}(s[q\backslash x]) \to (r[q\backslash y])\mathbf{V}(s[q\backslash x]))$
8. $(r[q\backslash x])\mathbf{V}(s[q\backslash x]) \to (r[q\backslash y])\mathbf{V}(s[q\backslash x])$ **R1-MP** 6, 7
9. $\mathbf{G}(s[q\backslash x] \to s[q\backslash y])$ **R2-G** 5
10. $\mathbf{G}(s[q\backslash x] \to s[q\backslash y]) \to$ **A5-GV**
    $((r[q\backslash y])\mathbf{V}(s[q\backslash x]) \to (r[q\backslash y])\mathbf{V}(s[q\backslash y]))$
11. $(r[q\backslash y])\mathbf{V}(s[q\backslash x]) \to (r[q\backslash y])\mathbf{V}(s[q\backslash y])$ **R1-MP** 9, 10
12. $(r[q\backslash x])\mathbf{V}(s[q\backslash x]) \to (r[q\backslash y])\mathbf{V}(s[q\backslash y])$ **HS** 8, 11
13. $r[q\backslash y] \to r[q\backslash x]$ **IFF-E** 2
14. $s[q\backslash y] \to s[q\backslash x]$ **IFF-E** 3
15. $\mathbf{G}(r[q\backslash y] \to r[q\backslash x])$ **R2-G** 13
16. $\mathbf{G}(r[q\backslash y] \to r[q\backslash x]) \to$ **A4-GV**
    $((r[q\backslash y])\mathbf{V}(s[q\backslash y]) \to (r[q\backslash x])\mathbf{V}(s[q\backslash y]))$
17. $(r[q\backslash y])\mathbf{V}(s[q\backslash y]) \to (r[q\backslash x])\mathbf{V}(s[q\backslash y])$ **R1-MP** 15, 16
18. $\mathbf{G}(s[q\backslash y] \to s[q\backslash x])$ **R2-G** 14
19. $\mathbf{G}(s[q\backslash y] \to s[q\backslash x]) \to$ **A5-GV**
    $((r[q\backslash x])\mathbf{V}(s[q\backslash y]) \to (r[q\backslash x])\mathbf{V}(s[q\backslash x]))$

20. $(r[q\backslash x])\mathbf{V}(s[q\backslash y]) \to (r[q\backslash x])\mathbf{V}(s[q\backslash x])$ **R1-MP** 18, 19
21. $(r[q\backslash y])\mathbf{V}(s[q\backslash y]) \to (r[q\backslash x])\mathbf{V}(s[q\backslash x])$ **HS** 17, 20
22. $(r[q\backslash x])\mathbf{V}(s[q\backslash x]) \leftrightarrow (r[q\backslash y])\mathbf{V}(s[q\backslash y])$ **IFF-LR** 12, 21
23. $(r\mathbf{V}s)[q\backslash x] \leftrightarrow (r\mathbf{V}s)[q\backslash y]$ **DEF** $[\cdot]$ 22

(DIST-ORV) $\vdash_{\Delta}^{PLTL} p\mathbf{V}r \vee q\mathbf{V}r \to (p \vee q)\mathbf{V}r$ (distribution of $\mathbf{V}$ over $\vee$)[2]

Proof:

1. $p \to p$ **REFL**
2. $p \to p \vee q$ **OR-R** 1
3. $\mathbf{G}(p \to p \vee q)$ **R2-G** 2
4. $\mathbf{G}(p \to p \vee q) \to (p\mathbf{V}r \to (p \vee q)\mathbf{V}r)$ **A4-GV**
5. $p\mathbf{V}r \to (p \vee q)\mathbf{V}r$ **R1-MP** 3, 4
6. $q \to q$ **REFL**
7. $q \to p \vee q$ **OR-R** 6
8. $\mathbf{G}(q \to p \vee q)$ **R2-G** 7
9. $\mathbf{G}(q \to p \vee q) \to (q\mathbf{V}r \to (p \vee q)\mathbf{V}r)$ **A4-GV**
10. $q\mathbf{V}r \to (p \vee q)\mathbf{V}r$ **R1-MP** 8, 9
11. $p\mathbf{V}r \vee q\mathbf{V}r \to (p \vee q)\mathbf{V}r$ **OR-L** 5, 10

(DIST-ANDV) $\vdash_{\Delta}^{PLTL} p\mathbf{V}(q \wedge r) \leftrightarrow p\mathbf{V}q \wedge p\mathbf{V}r$ (distribution of $\mathbf{V}$ over $\wedge$)

Proof:

1. $p\mathbf{V}q \wedge p\mathbf{V}r \to (p \wedge p)\mathbf{V}(q \wedge r) \vee (p \wedge r)\mathbf{V}(q \wedge r) \vee (q \wedge p)\mathbf{V}(q \wedge r)$ **A8-V**
2. $p \to p$ **REFL**
3. $p \wedge p \to p$ **AND-L** 2
4. $p \to p \wedge p$ **AND-R** 2, 2
5. $p \leftrightarrow p \wedge p$ **IFF-RL** 3, 4
6. $p\mathbf{V}(q \wedge r) \vee (p \wedge r)\mathbf{V}(q \wedge r) \vee (q \wedge p)\mathbf{V}(q \wedge r) \leftrightarrow$ **REPL-PLTL** 5
   $(p \wedge p)\mathbf{V}(q \wedge r) \vee (p \wedge r)\mathbf{V}(q \wedge r) \vee (q \wedge p)\mathbf{V}(q \wedge r)$
7. $(p \wedge p)\mathbf{V}(q \wedge r) \vee (p \wedge r)\mathbf{V}(q \wedge r) \vee (q \wedge p)\mathbf{V}(q \wedge r) \to$ **IFF-E** 6
   $p\mathbf{V}(q \wedge r) \vee (p \wedge r)\mathbf{V}(q \wedge r) \vee (q \wedge p)\mathbf{V}(q \wedge r)$
8. $p\mathbf{V}q \wedge p\mathbf{V}r \to p\mathbf{V}(q \wedge r) \vee (p \wedge r)\mathbf{V}(q \wedge r) \vee (q \wedge p)\mathbf{V}(q \wedge r)$ **HS** 1, 7
9. $p \wedge r \to p$ **AND-L** 2
10. $\mathbf{G}(p \wedge r \to p)$ **R2-G** 9
11. $\mathbf{G}(p \wedge r \to p) \to ((p \wedge r)\mathbf{V}(q \wedge r) \to p\mathbf{V}(q \wedge r))$ **A4-GV**
12. $(p \wedge r)\mathbf{V}(q \wedge r) \to p\mathbf{V}(q \wedge r)$ **R1-MP** 10, 11
13. $q \wedge p \to p$ **AND-L** 2
14. $\mathbf{G}(q \wedge p \to p)$ **R2-G** 13
15. $\mathbf{G}(q \wedge p \to p) \to ((q \wedge p)\mathbf{V}(q \wedge r) \to p\mathbf{V}(q \wedge r))$ **A4-GV**
16. $(q \wedge p)\mathbf{V}(q \wedge r) \to p\mathbf{V}(q \wedge r)$ **R1-MP** 14, 15
17. $(p \wedge r)\mathbf{V}(q \wedge r) \vee (q \wedge p)\mathbf{V}(q \wedge r) \to p\mathbf{V}(q \wedge r)$ **OR-L** 12, 16
18. $p\mathbf{V}(q \wedge r) \to p\mathbf{V}(q \wedge r)$ **REFL**
19. $p\mathbf{V}(q \wedge r) \vee (p \wedge r)\mathbf{V}(q \wedge r) \vee (q \wedge p)\mathbf{V}(q \wedge r) \to p\mathbf{V}(q \wedge r)$ **OR-L** 18, 17
20. $p\mathbf{V}q \wedge p\mathbf{V}r \to p\mathbf{V}(q \wedge r)$ **HS** 8, 19
21. $q \to q$ **REFL**
22. $q \wedge r \to q$ **AND-L** 21

[2]Note that the converse corresponds to **A9-V**.

23. $\mathbf{G}(q \wedge r \rightarrow q)$ **R2-G** 22
24. $\mathbf{G}(q \wedge r \rightarrow q) \rightarrow (p\mathbf{V}(q \wedge r) \rightarrow p\mathbf{V}q)$ **A5-GV**
25. $p\mathbf{V}(q \wedge r) \rightarrow p\mathbf{V}q$ **R1-MP** 23, 24
26. $r \rightarrow r$ **REFL**
27. $q \wedge r \rightarrow r$ **AND-L** 26
28. $\mathbf{G}(q \wedge r \rightarrow r)$ **R2-G** 27
29. $\mathbf{G}(q \wedge r \rightarrow r) \rightarrow (p\mathbf{V}(q \wedge r) \rightarrow p\mathbf{V}r)$ **A5-GV**
30. $p\mathbf{V}(q \wedge r) \rightarrow p\mathbf{V}r$ **R1-MP** 28, 29
31. $p\mathbf{V}(q \wedge r) \rightarrow p\mathbf{V}q \wedge p\mathbf{V}r$ **AND-R** 25, 30
32. $p\mathbf{V}(q \wedge r) \leftrightarrow p\mathbf{V}q \wedge p\mathbf{V}r$ **IFF-RL** 20, 31

(IDEM-F) $\vdash_{\Delta}^{PLTL}$ $\mathbf{FF}p \rightarrow \mathbf{F}p$ (idempotence of $\mathbf{F}$)[3]

Proof:

1. $p\mathbf{V}\top \rightarrow p\mathbf{V}\top$ **REFL**
2. $\top \wedge p\mathbf{V}\top \rightarrow p\mathbf{V}\top$ **AND-L** 1
3. $p\mathbf{V}\top \rightarrow (p\mathbf{V}\top \rightarrow p\mathbf{V}\top)$ **A1-I**
4. $p\mathbf{V}\top \rightarrow \top$ **D1-$\top$** 3
5. $p\mathbf{V}\top \rightarrow \top \wedge p\mathbf{V}\top$ **AND-R** 1, 4
6. $\top \wedge p\mathbf{V}\top \leftrightarrow p\mathbf{V}\top$ **IFF-RL** 2, 5
7. $(\top \wedge p\mathbf{V}\top)\mathbf{V}\top \leftrightarrow (p\mathbf{V}\top)\mathbf{V}\top$ **REPL-PLTL** 6
8. $(p\mathbf{V}\top)\mathbf{V}\top \rightarrow (\top \wedge p\mathbf{V}\top)\mathbf{V}\top$ **IFF-E** 7
9. $(\top \wedge p\mathbf{V}\top)\mathbf{V}\top \rightarrow p\mathbf{V}\top$ **A7-V**
10. $(p\mathbf{V}\top)\mathbf{V}\top \rightarrow p\mathbf{V}\top$ **HS** 8, 9
11. $p\mathbf{V}\top \vee (p\mathbf{V}\top)\mathbf{V}\top \rightarrow p\mathbf{V}\top$ **OR-L** 1, 10
12. $(p \vee p\mathbf{V}\top)\mathbf{V}\top \leftrightarrow p\mathbf{V}\top \vee (p\mathbf{V}\top)\mathbf{V}\top$ **DIST-ORV**
13. $(p \vee p\mathbf{V}\top)\mathbf{V}\top \rightarrow p\mathbf{V}\top \vee (p\mathbf{V}\top)\mathbf{V}\top$ **IFF-E** 12
14. $(p \vee p\mathbf{V}\top)\mathbf{V}\top \rightarrow p\mathbf{V}\top$ **HS** 13, 11
15. $p\mathbf{V}\top \vee (p \vee p\mathbf{V}\top)\mathbf{V}\top \rightarrow p\mathbf{V}\top$ **OR-L** 1, 14
16. $p\mathbf{V}\top \vee (p \vee p\mathbf{V}\top)\mathbf{V}\top \rightarrow p \vee p\mathbf{V}\top$ **OR-R** 15
17. $p \rightarrow p$ **REFL**
18. $p \rightarrow p \vee p\mathbf{V}\top$ **OR-R** 17
19. $p \vee p\mathbf{V}\top \vee (p \vee p\mathbf{V}\top)\mathbf{V}\top \rightarrow p \vee p\mathbf{V}\top$ **OR-R** 16, 18
20. $\mathbf{FF}p \rightarrow \mathbf{F}p$ **D8-F** 19

(IDEM-G) $\vdash_{\Delta}^{PLTL}$ $\mathbf{G}p \rightarrow \mathbf{GG}p$ (idempotence of $\mathbf{G}$)[4]

Proof:

1. $\mathbf{FF}(\neg p) \rightarrow \mathbf{F}(\neg p)$ **IDEM-F**
2. $\mathbf{F}(\neg\mathbf{G}(\neg\neg p)) \rightarrow \neg\mathbf{G}(\neg\neg p)$ **D9-G** 1
3. $\neg\neg\neg p \rightarrow \neg p$ **DOUB**
4. $(\neg\neg\neg p \rightarrow \neg p) \rightarrow (p \rightarrow \neg\neg p)$ **A3-N**
5. $p \rightarrow \neg\neg p$ **R1-MP** 3, 4
6. $\neg\neg p \rightarrow p$ **DOUB**
7. $p \leftrightarrow \neg\neg p$ **IFF-RL** 5, 6
8. $(\mathbf{F}(\neg\mathbf{G}(\neg\neg p)) \rightarrow \neg\mathbf{G}(\neg\neg p)) \leftrightarrow (\mathbf{F}(\neg\mathbf{G}p) \rightarrow \neg\mathbf{G}p)$ **REPL-PLTL** 7

[3]Note that the implication in the oposite direction is obtained from the definition of **F**.
[4]The same comment for **IDEM-F** applies here.

9. $(\mathbf{F}(\neg\mathbf{G}(\neg\neg p)) \to \neg\mathbf{G}(\neg\neg p)) \to (\mathbf{F}(\neg\mathbf{G}p) \to \neg\mathbf{G}p)$ **IFF-E** 8
10. $\mathbf{F}(\neg\mathbf{G}p) \to \neg\mathbf{G}p$ **R1-MP** 2, 9
11. $(\mathbf{F}(\neg\mathbf{G}p) \to \neg\mathbf{G}p) \to (\neg\neg\mathbf{G}p \to \neg\mathbf{F}(\neg\mathbf{G}p))$ **CONP**
12. $\neg\neg\mathbf{G}p \to \neg\mathbf{F}(\neg\mathbf{G}p)$ **R1-MP** 10, 11
13. $\neg\neg\neg\mathbf{G}p \to \neg\mathbf{G}p$ **DOUB**
14. $(\neg\neg\neg\mathbf{G}p \to \neg\mathbf{G}p) \to (\mathbf{G}p \to \neg\neg\mathbf{G}p)$ **A3-N**
15. $\mathbf{G}p \to \neg\neg\mathbf{G}p$ **R1-MP** 13, 14
16. $\mathbf{G}p \to \neg\mathbf{F}(\neg\mathbf{G}p)$ **HS** 15, 12
17. $\mathbf{G}p \to \mathbf{GG}p$ **D9-G** 16

(DUAL-GF) $\vdash^{PLTL}_{\Delta}$ $\mathbf{F}(\neg p) \leftrightarrow \neg\mathbf{G}p$ (duality between $\mathbf{G}$ and $\mathbf{F}$)

Proof:

1. $\neg(p \wedge \neg(\neg p)\mathbf{V}\top) \leftrightarrow \neg p \vee \neg\neg(\neg p)\mathbf{V}\top$ **DM**
2. $\neg p \vee \neg\neg(\neg p)\mathbf{V}\top \to \neg(p \wedge \neg(\neg p)\mathbf{V}\top)$ **IFF-E** 1
3. $\neg(p \wedge \neg(\neg p)\mathbf{V}\top) \to \neg p \vee \neg\neg(\neg p)\mathbf{V}\top$ **IFF-E** 1
4. $\neg\neg(\neg p)\mathbf{V}\top \to (\neg p)\mathbf{V}\top$ **DOUB**
5. $\neg\neg(\neg p)\mathbf{V}\top \to \neg p \vee (\neg p)\mathbf{V}\top$ **OR-L** 4
6. $\neg p \to \neg p$ **REFL**
7. $\neg p \to \neg p \vee (\neg p)\mathbf{V}\top$ **OR-L** 6
8. $\neg p \vee \neg\neg(\neg p)\mathbf{V}\top \to \neg p \vee (\neg p)\mathbf{V}\top$ **OR-R** 5, 7
9. $\neg(p \wedge \neg(\neg p)\mathbf{V}\top) \to \neg p \vee (\neg p)\mathbf{V}\top$ **HS** 3, 8
10. $\neg\neg\neg(\neg p)\mathbf{V}\top \to \neg(\neg p)\mathbf{V}\top$ **DOUB**
11. $(\neg\neg\neg(\neg p)\mathbf{V}\top \to \neg(\neg p)\mathbf{V}\top) \to ((\neg p)\mathbf{V}\top \to \neg\neg(\neg p)\mathbf{V}\top)$ **A3-N**
12. $(\neg p)\mathbf{V}\top \to \neg\neg(\neg p)\mathbf{V}\top$ **R1-MP** 10, 11
13. $(\neg p)\mathbf{V}\top \to \neg p \vee \neg\neg(\neg p)\mathbf{V}\top$ **OR-L** 12
14. $\neg p \to \neg p \vee \neg\neg(\neg p)\mathbf{V}\top$ **OR-L** 6
15. $\neg p \vee (\neg p)\mathbf{V}\top \to \neg p \vee \neg\neg(\neg p)\mathbf{V}\top$ **OR-R** 13, 14
16. $\neg p \vee (\neg p)\mathbf{V}\top \to \neg(p \wedge \neg(\neg p)\mathbf{V}\top)$ **HS** 15, 2
17. $\neg(p \wedge \neg(\neg p)\mathbf{V}\top) \leftrightarrow \neg p \vee (\neg p)\mathbf{V}\top$ **IFF-RL** 9, 16
18. $\mathbf{F}(\neg p) \leftrightarrow \neg\mathbf{G}p$ **D8-F**, **D9-G** 17

(REFL-G) $\vdash^{PLTL}_{\Delta}$ $\mathbf{G}p \to p$ (reflexivity of $\mathbf{G}$)

Proof:

1. $p \to p$ **REFL**
2. $p \wedge \neg(\neg p)\mathbf{V}\top \to p$ **AND-L** 1
3. $\mathbf{G}p \to p$ **D9-G**

(MON-G) $\vdash^{PLTL}_{\Delta}$ $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ (monotonicity of $\mathbf{G}$)

Proof:

1. $(p \to q) \to (\neg q \to \neg p)$ **CONP**
2. $(\neg q \to \neg p) \to (p \to q)$ **A3-N**
3. $(p \to q) \leftrightarrow (\neg q \to \neg p)$ **IFF-RL** 1, 2
4. $\mathbf{G}(p \to q) \leftrightarrow \mathbf{G}(\neg q \to \neg p)$ **REPL-PLTL** 3
5. $\mathbf{G}(p \to q) \to \mathbf{G}(\neg q \to \neg p)$ **IFF-E** 4
6. $\mathbf{G}(\neg q \to \neg p) \to ((\neg q)\mathbf{V}\top \to (\neg p)\mathbf{V}\top)$ **A5-GV**
7. $\mathbf{G}(p \to q) \to ((\neg q)\mathbf{V}\top \to (\neg p)\mathbf{V}\top)$ **HS** 5, 6

8. $((\neg q)\mathbf{V}\top \to (\neg p)\mathbf{V}\top) \to (\neg(\neg p)\mathbf{V}\top \to \neg(\neg q)\mathbf{V}\top)$ **CONP**
9. $\mathbf{G}(p \to q) \to (\neg(\neg p)\mathbf{V}\top \to \neg(\neg q)\mathbf{V}\top)$ **HS** 7, 8
10. $\neg(\neg p)\mathbf{V}\top \to \neg(\neg p)\mathbf{V}\top$ **REFL**
11. $p \wedge \neg(\neg p)\mathbf{V}\top \to \neg(\neg p)\mathbf{V}\top$ **AND-L** 10
12. $(p \wedge \neg(\neg p)\mathbf{V}\top \to \neg(\neg p)\mathbf{V}\top) \to$ $((\neg(\neg p)\mathbf{V}\top \to \neg(\neg q)\mathbf{V}\top) \to (p \wedge \neg(\neg p)\mathbf{V}\top \to \neg(\neg q)\mathbf{V}\top))$ **RTRAN**
13. $(\neg(\neg p)\mathbf{V}\top \to \neg(\neg q)\mathbf{V}\top) \to (p \wedge \neg(\neg p)\mathbf{V}\top \to \neg(\neg q)\mathbf{V}\top)$ **R1-MP** 11, 12
14. $\mathbf{G}(p \to q) \to (p \wedge \neg(\neg p)\mathbf{V}\top \to \neg(\neg q)\mathbf{V}\top)$ **HS** 9, 13
15. $\mathbf{G}(p \to q) \to (p \to q)$ **REFL-G**
16. $p \to p$ **REFL**
17. $p \wedge \neg(\neg p)\mathbf{V}\top \to p$ **AND-L** 16
18. $(p \wedge \neg(\neg p)\mathbf{V}\top \to p) \to ((p \to q) \to (p \wedge \neg(\neg p)\mathbf{V}\top \to q))$ **RTRAN**
19. $(p \to q) \to (p \wedge \neg(\neg p)\mathbf{V}\top \to q)$ **R1-MP** 17, 18
20. $\mathbf{G}(p \to q) \to (p \wedge \neg(\neg p)\mathbf{V}\top \to q)$ **HS** 15, 19
21. $\mathbf{G}(p \to q) \to$ $((p \wedge \neg(\neg p)\mathbf{V}\top \to q) \wedge (p \wedge \neg(\neg p)\mathbf{V}\top \to \neg(\neg q)\mathbf{V}\top))$ **AND-R** 14, 20
22. $((p \wedge \neg(\neg p)\mathbf{V}\top \to q) \wedge (p \wedge \neg(\neg p)\mathbf{V}\top \to \neg(\neg q)\mathbf{V}\top)) \leftrightarrow$ $(p \wedge \neg(\neg p)\mathbf{V}\top \to q \wedge \neg(\neg p)\mathbf{V}\top)$ **DIST-IA**
23. $((p \wedge \neg(\neg p)\mathbf{V}\top \to q) \wedge (p \wedge \neg(\neg p)\mathbf{V}\top \to \neg(\neg q)\mathbf{V}\top)) \to$ $(p \wedge \neg(\neg p)\mathbf{V}\top \to q \wedge \neg(\neg p)\mathbf{V}\top)$ **IFF-E** 22
24. $\mathbf{G}(p \to q) \to (p \wedge \neg(\neg p)\mathbf{V}\top \to q \wedge \neg(\neg p)\mathbf{V}\top)$ **HS** 21, 23
25. $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ **D9-G** 24

(RPL-GX) $\vdash_{\Delta}^{PLTL} \mathbf{G}p \to \mathbf{X}p$

Proof:

1. $p \to (\top \to p)$ **A1-I**
2. $p \to p$ **REFL**
3. $(p \to p) \to (((p \to p) \to p) \to p)$ **EXP**
4. $(\top \to p) \to p$ **R1-MP** 2, 3; **D1-**$\top$
5. $p \leftrightarrow (\top \to p)$ **IFF-RL** 1, 4
6. $\mathbf{G}p \leftrightarrow \mathbf{G}(\top \to p)$ **REPL-PLTL** 4
7. $\mathbf{G}p \to \mathbf{G}(\top \to p)$ **IFF-E** 6
8. $\mathbf{G}(\top \to p) \to (\mathbf{X}\top \to \mathbf{X}p)$ **A5-GV**, **D6-X**
9. $(\mathbf{G}(\top \to p) \to (\mathbf{X}\top \to \mathbf{X}p)) \to (\mathbf{X}\top \to (\mathbf{G}(\top \to p) \to \mathbf{X}p))$ **PERM**
10. $\mathbf{X}\top \to (\mathbf{G}(\top \to p) \to \mathbf{X}p)$ **R1-MP** 8, 9
11. $\mathbf{X}\top$ **A11-X**
12. $\mathbf{G}(\top \to p) \to \mathbf{X}p$ **R1-MP** 10, 11
13. $\mathbf{G}p \to \mathbf{X}p$ **HS** 7, 12

(EXP-GX) $\vdash_{\Delta}^{PLTL} \mathbf{G}p \to \mathbf{XG}p$

Proof:

1. $\mathbf{G}p \to \mathbf{GG}p$ **IDEM-G**
2. $\mathbf{GG}p \to \mathbf{XG}p$ **REPL-GX**
3. $\mathbf{G}p \to \mathbf{XG}p$ **HS** 1, 2

(G⊤) $\vdash^{PLTL}_{\Delta}$ $\mathbf{G}\top$

Proof:

1. $\mathbf{GX}\top$ **A11-X**, **R2-G**
2. $\mathbf{XG}\top$ **COM-GX**, **IFF-E**, **R1-MP** 1
3. $\top \wedge \mathbf{XG}\top$ **REFL**, **D1-⊤**, **AND-I** 2
4. $\mathbf{G}\top$ **FIX-G**, **IFF-E**, **R1-MP** 3

(NEG-V⊤) $\vdash^{PLTL}_{\Delta}$ $\neg(\bot\mathbf{V}\top)$

Proof:

1. $p \to p$ **REFL**
2. $\mathbf{G}\top$ **D1-⊤**, **R2-G** 1
3. $p \wedge \neg(\neg\top)\mathbf{V}\top$ **D9-G** 2
4. $\neg\bot\mathbf{V}\top$ **D2-⊥**, **AND-E** 3

(NEG-V⊥) $\vdash^{PLTL}_{\Delta}$ $\neg(\bot\mathbf{V}\bot)$

Proof:

1. $p \to p$ **REFL**
2. $(p \to p) \to (\neg(p \to p) \to (p \to p))$ **A1-I**
3. $\neg(p \to p) \to (p \to p)$ **R1-MP** 1, 2
4. $\bot \to \top$ **D1-⊤**, **D2-⊥** 3
5. $\mathbf{G}(\bot \to \top)$ **R2-G** 4
6. $\mathbf{G}(\bot \to \top) \to (\bot\mathbf{V}\bot \to \bot\mathbf{V}\top)$ **A5-GV**
7. $\bot\mathbf{V}\bot \to \bot\mathbf{V}\top$ **R1-MP** 5, 6
8. $(\bot\mathbf{V}\bot \to \bot\mathbf{V}\top) \to (\neg\bot\mathbf{V}\top \to \neg\bot\mathbf{V}\bot)$ **CONP**
9. $\neg\bot\mathbf{V}\top \to \neg\bot\mathbf{V}\bot$ **R1-MP** 7, 8
10. $\neg\bot\mathbf{V}\top$ **NEG-V⊤**
11. $\neg\bot\mathbf{V}\bot$ **R1-MP** 10, 9

(FUN-X) $\vdash^{PLTL}_{\Delta}$ $\neg\mathbf{X}p \leftrightarrow \mathbf{X}(\neg p)$ (functionality of $\mathbf{X}$)

Proof:

1. $\mathbf{X}(p \vee \neg p) \leftrightarrow (\mathbf{X}p \vee \mathbf{X}(\neg p))$ **DIST-ORV**, **D6-X**
2. $\mathbf{X}(\neg p \to \neg p) \to (\neg\mathbf{X}p \to \mathbf{X}(\neg p))$ **D3-OR**, **IFF-E** 1
3. $\neg p \to \neg p$ **REFL**
4. $\mathbf{G}(\neg p \to \neg p)$ **R2-G** 3
5. $\mathbf{G}(\neg p \to \neg p) \to \mathbf{X}(\neg p \to \neg p)$ **REPL-GX**
6. $\mathbf{X}(\neg p \to \neg p)$ **R1-MP** 4, 5
7. $\mathbf{X}(\neg p) \to \neg\mathbf{X}p$ **R1-MP** 6, 2
8. $(\neg p)\mathbf{V}\bot \wedge p\mathbf{V}\bot \to$ **A8-V**
   $(\neg p \wedge p)\mathbf{V}(\bot \wedge \bot) \vee (\neg p \wedge \bot)\mathbf{V}(\bot \wedge \bot) \vee (p \wedge \bot)\mathbf{V}(\bot \wedge \bot)$
9. $\bot \to \bot$ **REFL**
10. $\bot \wedge \bot \to \bot$ **AND-L** 9
11. $\bot \to \bot \wedge \bot$ **AND-R** 9, 9
12. $\bot \leftrightarrow \bot \wedge \bot$ **IFF-RL** 10, 11
13. $(\neg p \wedge p)\mathbf{V}\bot \vee (\neg p \wedge \bot)\mathbf{V}\bot \vee (p \wedge \bot)\mathbf{V}\bot \leftrightarrow$ **REPL-PLTL** 12
    $(\neg p \wedge p)\mathbf{V}(\bot \wedge \bot) \vee (\neg p \wedge \bot)\mathbf{V}(\bot \wedge \bot) \vee (p \wedge \bot)\mathbf{V}(\bot \wedge \bot)$
14. $(\neg p)\mathbf{V}\bot \wedge p\mathbf{V}\bot \to$ **IFF-E** 13, **HS** 8

	$(\neg p \wedge p)\mathbf{V}\bot \vee (\neg p \wedge \bot)\mathbf{V}\bot \vee (p \wedge \bot)\mathbf{V}\bot$	
15.	$\mathbf{X}(\neg p) \wedge \mathbf{X}p \to \mathbf{X}(\neg p \wedge p) \vee \mathbf{X}(\neg p \wedge \bot) \vee \mathbf{X}(p \wedge \bot)$	**D6-X** 14
16.	$\neg p \wedge \bot \to \bot$	**AND-L** 9
17.	$(\neg p \to \neg p) \to (\neg(\neg p \to \neg p) \to \neg p)$	**NEG-L**
18.	$\bot \to \neg p$	**R1-MP** 3, 17; **D2-**$\bot$
19.	$\bot \to \neg p \wedge \bot$	**AND-R** 9, 18
20.	$\bot \leftrightarrow \neg p \wedge \bot$	**IFF-RL** 16, 19
21.	$\mathbf{X}(\neg p \wedge p) \vee \mathbf{X}\bot \vee \mathbf{X}(p \wedge \bot) \leftrightarrow$ $\mathbf{X}(\neg p \wedge p) \vee \mathbf{X}(\neg p \wedge \bot) \vee \mathbf{X}(p \wedge \bot)$	**REPL-PLTL** 20
22.	$\mathbf{X}(\neg p)\bot \wedge \mathbf{X}p \to \mathbf{X}(\neg p \wedge p) \vee \mathbf{X}\bot \vee \mathbf{X}(p \wedge \bot)$	**IFF-E** 21, **HS** 15
23.	$p \wedge \bot \to \bot$	**AND-L** 9
24.	$(p \to p) \to (\neg(p \to p) \to p)$	**NEG-L**
25.	$\bot \to p$	**R1-MP** 3, 24; **D2-**$\bot$
26.	$\bot \to p \wedge \bot$	**AND-R** 9, 25
27.	$\bot \leftrightarrow \neg p \wedge \bot$	**IFF-RL** 23, 26
28.	$\mathbf{X}(\neg p \wedge p) \vee \mathbf{X}\bot \vee \mathbf{X}(p \wedge \bot) \leftrightarrow$ $\mathbf{X}(\neg p \wedge p) \vee \mathbf{X}\bot \vee \mathbf{X}\bot$	**REPL-PLTL** 27
29.	$\mathbf{X}(\neg p) \wedge \mathbf{X}p \to \mathbf{X}(\neg p \wedge p) \vee \mathbf{X}\bot \vee \mathbf{X}\bot$	**IFF-E** 28, **HS** 22
30.	$\bot \to \neg p \wedge p$	**AND-R** 18, 25
31.	$\neg(\neg p \to \neg p) \to \neg(\neg p \to \neg p)$	**REFL**
32.	$(\neg p \wedge p) \to \bot$	**D4-AND**, **D2-**$\bot$ 31
33.	$\bot \leftrightarrow \neg p \wedge p$	**IFF-RL** 30, 32
34.	$\mathbf{X}(\neg p \wedge p) \vee \mathbf{X}\bot \vee \mathbf{X}\bot \leftrightarrow \mathbf{X}\bot \vee \mathbf{X}\bot \vee \mathbf{X}\bot$	**REPL-PLTL** 33
35.	$\mathbf{X}(\neg p) \wedge \mathbf{X}p \to \mathbf{X}\bot \vee \mathbf{X}\bot \vee \mathbf{X}\bot$	**IFF-E** 34, **HS** 29
36.	$\mathbf{X}\bot \to \mathbf{X}\bot$	**REFL**
37.	$\mathbf{X}\bot \vee \mathbf{X}\bot \to \mathbf{X}\bot$	**OR-L** 36, 36
38.	$\mathbf{X}\bot \vee \mathbf{X}\bot \vee \mathbf{X}\bot \to \mathbf{X}\bot$	**OR-L** 36, 37
39.	$\mathbf{X}(\neg p) \wedge \mathbf{X}p \to \mathbf{X}\bot$	**HS** 35, 38
40.	$(\mathbf{X}(\neg p) \wedge \mathbf{X}p \to \mathbf{X}\bot) \to (\neg\mathbf{X}\bot \to \neg(\mathbf{X}(\neg p) \wedge \mathbf{X}p))$	**CONP**
41.	$\neg\mathbf{X}\bot \to \neg(\mathbf{X}(\neg p) \wedge \mathbf{X}p)$	**R1-MP** 39, 40
42.	$\neg\mathbf{X}\bot$	**NEG-V**$\top$
43.	$\neg(\mathbf{X}(\neg p) \wedge \mathbf{X}p)$	**R1-MP** 41, 42
44.	$\neg\neg(\mathbf{X}(\neg p) \to \neg\mathbf{X}p)$	**D4-AND** 43
45.	$\neg\neg(\mathbf{X}(\neg p) \to \neg\mathbf{X}p) \to (\mathbf{X}(\neg p) \to \neg\mathbf{X}p)$	**DOUB**
46.	$\mathbf{X}(\neg p) \to \neg\mathbf{X}p$	**R1-MP** 44, 45
47.	$\neg\mathbf{X}p \leftrightarrow \mathbf{X}(\neg p)$	**IFF-RL** 7, 46

(MON-X) $\vdash^{PLTL}_{\Delta}$ $\mathbf{X}(p \to q) \to (\mathbf{X}p \to \mathbf{X}q)$ (monotonicity of $\mathbf{X}$)

Proof:

1.	$(\neg p \vee q)\mathbf{V}\bot \leftrightarrow ((\neg p)\mathbf{V}\bot \vee q\mathbf{V}\bot)$	**DIST-ORV**
2.	$\mathbf{X}(\neg\neg p \to q) \leftrightarrow (\neg\mathbf{X}(\neg p) \to \mathbf{X}q)$	**D6-X**, **D3-OR** 1
3.	$\mathbf{X}(\neg\neg p \to q) \to (\neg\mathbf{X}(\neg p) \to \mathbf{X}q)$	**IFF-E** 2
4.	$\neg\neg p \to p$	**DOUB**
5.	$\neg\neg\neg p \to \neg p$	**DOUB**
6.	$(\neg\neg\neg p \to \neg p) \to (p \to \neg\neg p)$	**A3-N**
7.	$p \to \neg\neg p$	**R1-MP** 5, 6
8.	$p \leftrightarrow \neg\neg p$	**IFF-RL** 4, 7
9.	$\mathbf{X}(p \to q) \leftrightarrow \mathbf{X}(\neg\neg p \to q)$	**REPL-PLTL** 8

10. $\mathbf{X}(p \to q) \to \mathbf{X}(\neg\neg p \to q)$ **IFF-E** 9
11. $\mathbf{X}(p \to q) \to (\neg\mathbf{X}(\neg p) \to \mathbf{X}q)$ **HS** 10, 3
12. $\neg\mathbf{X}p \leftrightarrow \mathbf{X}(\neg p)$ **FUN-X**
13. $\mathbf{X}(\neg p) \to \neg\mathbf{X}p$ **IFF-E** 12
14. $(\mathbf{X}(\neg p) \to \neg\mathbf{X}p) \to (\neg\neg\mathbf{X}p \to \neg\mathbf{X}(\neg p))$ **CONP**
15. $\neg\neg\mathbf{X}p \to \neg\mathbf{X}(\neg p)$ **R1-MP** 13, 16
16. $\neg\neg\neg\mathbf{X}p \to \neg\mathbf{X}p$ **DOUB**
17. $(\neg\neg\neg\mathbf{X}p \to \neg\mathbf{X}p) \to (\mathbf{X}p \to \neg\neg\mathbf{X}p)$ **A3-N**
18. $\mathbf{X}p \to \neg\neg\mathbf{X}p$ **R1-MP** 16, 17
19. $\mathbf{X}p \to \neg\mathbf{X}(\neg p)$ **HS** 18, 15
20. $(\mathbf{X}p \to \neg\mathbf{X}(\neg p)) \to ((\neg\mathbf{X}(\neg p) \to \mathbf{X}q) \to (\mathbf{X}p \to \mathbf{X}q))$ **RTRAN**
21. $(\neg\mathbf{X}(\neg p) \to \mathbf{X}q) \to (\mathbf{X}p \to \mathbf{X}q)$ **R1-MP** 19, 20
22. $\mathbf{X}(p \to q) \to (\mathbf{X}p \to \mathbf{X}q)$ **HS** 11, 21

(MON-GX) $\vdash_{\Delta}^{PLTL}$ $\mathbf{G}(p \to q) \to (\mathbf{X}p \to \mathbf{X}q)$

Proof:

1. $\mathbf{G}(p \to q) \to \mathbf{X}(p \to q)$ **RPL-GX**
2. $\mathbf{X}(p \to q) \to (\mathbf{X}p \to \mathbf{X}q)$ **MON-X**
3. $\mathbf{G}(p \to q) \to (\mathbf{X}p \to \mathbf{X}q)$ **HS** 1, 2

(DIST-ANDX) $\vdash_{\Delta}^{PLTL}$ $\mathbf{X}(p \wedge q) \leftrightarrow \mathbf{X}p \wedge \mathbf{X}q$ (distribution of $\mathbf{X}$ over $\wedge$)

Proof:

1. $\mathbf{X}(p \to \neg q) \to (\mathbf{X}p \to \mathbf{X}(\neg q))$ **MON-X**
2. $\mathbf{X}(\neg q) \leftrightarrow \neg\mathbf{X}q$ **FUN-X**
3. $\mathbf{X}(\neg q) \to \neg\mathbf{X}q$ **IFF-E** 2
4. $(\mathbf{X}(\neg q) \to \neg\mathbf{X}q) \to ((\mathbf{X}p \to \mathbf{X}(\neg q)) \to (\mathbf{X}p \to \neg\mathbf{X}q))$ **LTRAN**
5. $(\mathbf{X}p \to \mathbf{X}(\neg q)) \to (\mathbf{X}p \to \neg\mathbf{X}q)$ **R1-MP** 3, 4
6. $\mathbf{X}(p \to \neg q) \to (\mathbf{X}p \to \neg\mathbf{X}q)$ **HS** 1, 5
7. $(\mathbf{X}(p \to \neg q) \to (\mathbf{X}p \to \neg\mathbf{X}q)) \to$ **CONP**
   $(\neg(\mathbf{X}p \to \neg\mathbf{X}q) \to \neg\mathbf{X}(p \to \neg q))$
8. $\neg(\mathbf{X}p \to \neg\mathbf{X}q) \to \neg\mathbf{X}(p \to \neg q)$ **R1-MP** 6, 7
9. $\neg\mathbf{X}(p \to \neg q) \leftrightarrow \mathbf{X}(\neg(p \to \neg q))$ **FUN-X**
10. $\neg\mathbf{X}(p \to \neg q) \to \mathbf{X}(\neg(p \to \neg q))$ **IFF-E** 9
11. $\neg(\mathbf{X}p \to \neg\mathbf{X}q) \to \mathbf{X}(\neg(p \to \neg q))$ **HS** 8, 10
12. $\mathbf{X}p \wedge \mathbf{X}q \to \mathbf{X}(p \wedge q)$ **D4-AND** 11
13. $(\neg p \vee \neg q)\mathbf{V}\bot \leftrightarrow (\neg p)\mathbf{V}\bot \vee (\neg q)\mathbf{V}\bot$ **DIST-ORV**
14. $\mathbf{X}(\neg p \vee \neg q) \leftrightarrow \mathbf{X}(\neg p) \vee \mathbf{X}(\neg q)$ **D6-X** 13
15. $\mathbf{X}(\neg p) \vee \mathbf{X}(\neg q) \to \mathbf{X}(\neg p \vee \neg q)$ **IFF-E** 14
16. $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$ **DM**
17. $\mathbf{X}(\neg p \vee \neg q) \leftrightarrow \mathbf{X}(\neg(p \wedge q))$ **REPL-PLTL** 15
18. $\mathbf{X}(\neg p \vee \neg q) \to \mathbf{X}(\neg(p \wedge q))$ **IFF-E** 17
19. $\mathbf{X}(\neg p) \vee \mathbf{X}(\neg q) \to \mathbf{X}(\neg(p \wedge q))$ **HS** 15, 18
20. $\mathbf{X}(\neg(p \wedge q)) \leftrightarrow \neg\mathbf{X}(p \wedge q)$ **FUN-X**
21. $\mathbf{X}(\neg(p \wedge q)) \to \neg\mathbf{X}(p \wedge q)$ **IFF-E** 20
22. $\mathbf{X}(\neg p) \vee \mathbf{X}(\neg q) \to \neg\mathbf{X}(p \wedge q)$ **HS** 19, 21
23. $(\neg\mathbf{X}(\neg p) \to \mathbf{X}(\neg q)) \to \neg\mathbf{X}(p \wedge q)$ **D3-OR** 22
24. $\mathbf{X}(\neg p) \leftrightarrow \neg\mathbf{X}p$ **FUN-X**

25. $\neg\mathbf{X}p \to \mathbf{X}(\neg p)$ **IFF-E** 24
26. $(\neg\mathbf{X}p \to \mathbf{X}(\neg p)) \to (\neg\mathbf{X}(\neg p) \to \neg\neg\mathbf{X}p)$ **CONP**
27. $\neg\mathbf{X}(\neg p) \to \neg\neg\mathbf{X}p$ **R1-MP** 25, 26
28. $\neg\neg\mathbf{X}p \to \mathbf{X}p$ **DOUB**
29. $\neg\mathbf{X}(\neg p) \to \mathbf{X}p$ **HS** 27, 28
30. $(\neg\mathbf{X}(\neg p) \to \mathbf{X}p) \to ((\mathbf{X}p \to \mathbf{X}(\neg q)) \to (\neg\mathbf{X}(\neg p) \to \mathbf{X}(\neg q)))$ **RTRAN**
31. $(\mathbf{X}p \to \mathbf{X}(\neg q)) \to (\neg\mathbf{X}(\neg p) \to \mathbf{X}(\neg q))$ **R1-MP** 29, 30
32. $(\mathbf{X}p \to \mathbf{X}(\neg q)) \to \neg\mathbf{X}(p \wedge q)$ **HS** 31, 23
33. $\neg\mathbf{X}q \to \mathbf{X}(\neg q)$ **IFF-E** 2
34. $(\neg\mathbf{X}q \to \mathbf{X}(\neg q)) \to ((\mathbf{X}p \to \neg\mathbf{X}q) \to (\mathbf{X}p \to \mathbf{X}(\neg q)))$ **LTRAN**
35. $(\mathbf{X}p \to \neg\mathbf{X}q) \to (\mathbf{X}p \to \mathbf{X}(\neg q))$ **R1-MP** 33, 34
36. $(\mathbf{X}p \to \neg\mathbf{X}q) \to \neg\mathbf{X}(p \wedge q)$ **HS** 35, 32
37. $((\mathbf{X}p \to \neg\mathbf{X}q) \to \neg\mathbf{X}(p \wedge q)) \to$ **CONP**
$(\neg\neg\mathbf{X}(p \wedge q) \to \neg(\mathbf{X}p \to \neg\mathbf{X}q))$
38. $\neg\neg\mathbf{X}(p \wedge q) \to \neg(\mathbf{X}p \to \neg\mathbf{X}q)$ **R1-MP** 36, 37
39. $\neg\neg\neg\mathbf{X}(p \wedge q) \to \neg\mathbf{X}(p \wedge q)$ **DOUB**
40. $(\neg\neg\neg\mathbf{X}(p \wedge q) \to \neg\mathbf{X}(p \wedge q)) \to (\mathbf{X}(p \wedge q) \to \neg\neg\mathbf{X}(p \wedge q))$ **A3-N**
41. $\mathbf{X}(p \wedge q) \to \neg\neg\mathbf{X}(p \wedge q)$ **R1-MP** 39, 40
42. $\mathbf{X}(p \wedge q) \to \neg(\mathbf{X}p \to \neg\mathbf{X}q)$ **HS** 41, 38
43. $\mathbf{X}(p \wedge q) \to \mathbf{X}p \wedge \mathbf{X}q$ **D4-AND** 42
44. $\mathbf{X}(p \wedge q) \leftrightarrow \mathbf{X}p \wedge \mathbf{X}q$ **IFF-RL** 43, 12

(FIX-V) $\vdash_{\Delta}^{PLTL} q\mathbf{V}p \leftrightarrow \mathbf{X}(q \vee p \wedge q\mathbf{V}p)$ (fixed point of $\mathbf{V}$)

Proof:

1. $p \to p$ **REFL**
2. $(p \to p) \to (\neg(p \to p) \to p)$ **NEG-L**
3. $\neg(p \to p) \to p$ **R1-MP** 1, 2
4. $\bot \to p$ **D2-$\bot$** 3
5. $\mathbf{G}(\bot \to p)$ **R2-G** 4
6. $\mathbf{G}(\bot \to p) \to (q\mathbf{V}\bot \to q\mathbf{V}p)$ **A5-GV**
7. $q\mathbf{V}\bot \to q\mathbf{V}p$ **R1-MP** 5, 6
8. $\mathbf{G}(\bot \to p) \to ((p \wedge q\mathbf{V}p)\mathbf{V}\bot \to (p \wedge q\mathbf{V}p)\mathbf{V}p)$ **A5-GV**
9. $(p \wedge q\mathbf{V}p)\mathbf{V}\bot \to (p \wedge q\mathbf{V}p)\mathbf{V}p$ **R1-MP** 5, 8
10. $(p \wedge q\mathbf{V}p)\mathbf{V}p \to q\mathbf{V}p$ **A7-V**
11. $(p \wedge q\mathbf{V}p)\mathbf{V}\bot \to q\mathbf{V}p$ **HS** 9, 10
12. $q\mathbf{V}\bot \vee (p \wedge q\mathbf{V}p)\mathbf{V}\bot \to q\mathbf{V}p$ **OR-L** 7, 11
13. $(q \vee (p \wedge q\mathbf{V}p))\mathbf{V}\bot \leftrightarrow q\mathbf{V}\bot \vee (p \wedge q\mathbf{V}p)\mathbf{V}\bot$ **DIST-ORV**
14. $(q \vee (p \wedge q\mathbf{V}p))\mathbf{V}\bot \to q\mathbf{V}\bot \vee (p \wedge q\mathbf{V}p)\mathbf{V}\bot$ **IFF-E** 13
15. $(q \vee (p \wedge q\mathbf{V}p))\mathbf{V}\bot \to q\mathbf{V}p$ **HS** 14, 12
16. $\mathbf{X}(q \vee (p \wedge q\mathbf{V}p)) \to q\mathbf{V}p$ **D6-X** 15
17. $q \to q$ **REFL**
18. $\mathbf{X}\top$ **A11-X**
19. $\top\mathbf{V}\bot$ **D6-X** 18
20. $q\mathbf{V}p \to q\mathbf{V}(p \wedge q\mathbf{V}p)$ **A6-V**
21. $q\mathbf{V}(p \wedge q\mathbf{V}p) \leftrightarrow q\mathbf{V}p \wedge q\mathbf{V}(q\mathbf{V}p)$ **DIST-ANDV**
22. $q\mathbf{V}(p \wedge q\mathbf{V}p) \to q\mathbf{V}p \wedge q\mathbf{V}(q\mathbf{V}p)$ **IFF-E** 21
23. $q\mathbf{V}p \to q\mathbf{V}p \wedge q\mathbf{V}(q\mathbf{V}p)$ **HS** 20,22
24. $q\mathbf{V}(q\mathbf{V}p) \to q\mathbf{V}(q\mathbf{V}p)$ **REFL**

25. $q\mathbf{V}p \wedge q\mathbf{V}(q\mathbf{V}p) \to q\mathbf{V}(q\mathbf{V}p)$ **AND-L** 24
26. $q\mathbf{V}p \to q\mathbf{V}(q\mathbf{V}p)$ **HS** 23, 25
27. $\top\mathbf{V}\bot \to (q\mathbf{V}(q\mathbf{V}p) \to \top\mathbf{V}\bot)$ **A1-I**
28. $q\mathbf{V}(q\mathbf{V}p) \to \top\mathbf{V}\bot$ **R1-MP** 19, 27
29. $q\mathbf{V}(q\mathbf{V}p) \to q\mathbf{V}(q\mathbf{V}p) \wedge \top\mathbf{V}\bot$ **AND-R** 24, 28
30. $q\mathbf{V}p \to q\mathbf{V}(q\mathbf{V}p) \wedge \top\mathbf{V}\bot$ **HS** 28, 29
31. $q\mathbf{V}(q\mathbf{V}p) \wedge \top\mathbf{V}\bot \to$ **A8-V**
$(q \wedge \top)\mathbf{V}(q\mathbf{V}p \wedge \bot) \vee (q \wedge \bot)\mathbf{V}(q\mathbf{V}p \wedge \bot) \vee (q\mathbf{V}p \wedge \top)\mathbf{V}(q\mathbf{V}p \wedge \bot)$
32. $\bot \to \bot$ **REFL**
33. $\bot \wedge q\mathbf{V}p \to \bot$ **AND-L** 33
34. $\bot \to \bot \wedge q\mathbf{V}p$ **AND-R** 4, 32
35. $\bot \wedge q\mathbf{V}p \leftrightarrow \bot$ **IFF-RL** 33, 34
36. $(q \wedge \top)\mathbf{V}\bot \vee (q \wedge \bot)\mathbf{V}\bot \vee (q\mathbf{V}p \wedge \top)\mathbf{V}\bot \leftrightarrow$ **REPL-PLTL** 35
$(q \wedge \top)\mathbf{V}(q\mathbf{V}p \wedge \bot) \vee (q \wedge \bot)\mathbf{V}(q\mathbf{V}p \wedge \bot) \vee (q\mathbf{V}p \wedge \top)\mathbf{V}(q\mathbf{V}p \wedge \bot)$
37. $q\mathbf{V}(q\mathbf{V}p) \wedge \top\mathbf{V}\bot \to$ **IFF-E** 36, **HS** 29
$(q \wedge \top)\mathbf{V}\bot \vee (q \wedge \bot)\mathbf{V}\bot \vee (q\mathbf{V}p \wedge \top)\mathbf{V}\bot$
38. $q\mathbf{V}(q\mathbf{V}p) \to (q \wedge \top)\mathbf{V}\bot \vee (q \wedge \bot)\mathbf{V}\bot \vee (q\mathbf{V}p \wedge \top)\mathbf{V}\bot$ **HS** 31, 37
39. $q\mathbf{V}p \to (q \wedge \top)\mathbf{V}\bot \vee (q \wedge \bot)\mathbf{V}\bot \vee (q\mathbf{V}p \wedge \top)\mathbf{V}\bot$ **HS** 26, 38
40. $(q \wedge \top)\mathbf{V}\bot \vee (q \wedge \bot)\mathbf{V}\bot \vee (q\mathbf{V}p \wedge \top)\mathbf{V}\bot \leftrightarrow$ **DIST-ORV**
$((q \wedge \top) \vee (q \wedge \bot) \vee (q\mathbf{V}p \wedge \top))\mathbf{V}\bot$
41. $(q \wedge \top)\mathbf{V}\bot \vee (q \wedge \bot)\mathbf{V}\bot \vee (q\mathbf{V}p \wedge \top)\mathbf{V}\bot \to$ **IFF-E** 40
$((q \wedge \top) \vee (q \wedge \bot) \vee (q\mathbf{V}p \wedge \top))\mathbf{V}\bot$
42. $q\mathbf{V}p \to ((q \wedge \top) \vee (q \wedge \bot) \vee (q\mathbf{V}p \wedge \top))\mathbf{V}\bot$ **HS** 39, 41
43. $q\mathbf{V}p \to \mathbf{X}((q \wedge \top) \vee (q \wedge \bot) \vee (q\mathbf{V}p \wedge \top))$ **D6-X** 42
44. $q\mathbf{V}p \to q\mathbf{V}p$ **REFL**
45. $\top\mathbf{V}\bot \to (q\mathbf{V}p \to \top\mathbf{V}\bot)$ **A1-I**
46. $q\mathbf{V}p \to \top\mathbf{V}\bot$ **R1-MP** 19, 45
47. $q\mathbf{V}p \to q\mathbf{V}p \wedge \top\mathbf{V}\bot$ **AND-R** 44, 46
48. $q\mathbf{V}p \wedge \top\mathbf{V}\bot \to$ **A8-V**
$(q \wedge \top)\mathbf{V}(p \wedge \bot) \vee (q \wedge \bot)\mathbf{V}(p \wedge \bot) \vee (p \wedge \top)\mathbf{V}(p \wedge \bot)$
49. $\bot \wedge p \to \bot$ **AND-L** 32
50. $\bot \to \bot \wedge p$ **AND-R** 4, 32
51. $\bot \leftrightarrow \bot \wedge p$ **IFF-RL** 48, 49
52. $(q \wedge \top)\mathbf{V}\bot \vee (q \wedge \bot)\mathbf{V}\bot \vee (p \wedge \top)\mathbf{V}\bot \leftrightarrow$ **REPL-PLTL** 51
$(q \wedge \top)\mathbf{V}(p \wedge \bot) \vee (q \wedge \bot)\mathbf{V}(p \wedge \bot) \vee (p \wedge \top)\mathbf{V}(p \wedge \bot)$
53. $(q \wedge \top)\mathbf{V}(p \wedge \bot) \vee (q \wedge \bot)\mathbf{V}(p \wedge \bot) \vee (p \wedge \top)\mathbf{V}(p \wedge \bot) \to$ **IFF-E** 52
$(q \wedge \top)\mathbf{V}\bot \vee (q \wedge \bot)\mathbf{V}\bot \vee (p \wedge \top)\mathbf{V}\bot$
54. $q\mathbf{V}p \wedge \top\mathbf{V}\bot \to (q \wedge \top)\mathbf{V}\bot \vee (q \wedge \bot)\mathbf{V}\bot \vee (p \wedge \top)\mathbf{V}\bot$ **HS** 48, 53
55. $q\mathbf{V}p \to (q \wedge \top)\mathbf{V}\bot \vee (q \wedge \bot)\mathbf{V}\bot \vee (p \wedge \top)\mathbf{V}\bot$ **HS** 47, 54
56. $(q \wedge \top)\mathbf{V}\bot \vee (q \wedge \bot)\mathbf{V}\bot \vee (p \wedge \top)\mathbf{V}\bot \leftrightarrow$ **DIST-ORV**
$((q \wedge \top) \vee (q \wedge \bot) \vee (p \wedge \top))\mathbf{V}$
57. $(q \wedge \top)\mathbf{V}\bot \vee (q \wedge \bot)\mathbf{V}\bot \vee (p \wedge \top)\mathbf{V}\bot \to$ **IFF-E** 56
$((q \wedge \top) \vee (q \wedge \bot) \vee (p \wedge \top))\mathbf{V}$
58. $q\mathbf{V}p \to ((q \wedge \top) \vee (q \wedge \bot) \vee (p \wedge \top))\mathbf{V}\bot$ **HS** 55, 57
59. $q\mathbf{V}p \to \mathbf{X}((q \wedge \top) \vee (q \wedge \bot) \vee (p \wedge \top))$ **D6-X** 58
60. $q\mathbf{V}p \to$ **AND-R** 43, 59
$\mathbf{X}((q \wedge \top) \vee (q \wedge \bot) \vee (q\mathbf{V}p \wedge \top)) \wedge \mathbf{X}((q \wedge \top) \vee (q \wedge \bot) \vee (p \wedge \top))$
61. $q\mathbf{V}p \to$ **DIST-ANDX**, **HS** 60

$\mathbf{X}(((q \wedge \top) \vee (q \wedge \bot) \vee (q\mathbf{V}p \wedge \top)) \wedge ((q \wedge \top) \vee (q \wedge \bot) \vee (p \wedge \top)))$

62. $q \wedge \top \to q$ **AND-L** 17
63. $q \to (q \to q)$ **A1-I**
64. $q \to \top$ **D1-**$\top$ 63
65. $q \to q \wedge \top$ **AND-R** 17, 64
66. $q \leftrightarrow q \wedge \top$ **IFF-RL** 62, 65
67. $\mathbf{X}((q \vee (q \wedge \bot) \vee (q\mathbf{V}p \wedge \top)) \wedge (q \vee (q \wedge \bot) \vee (p \wedge \top))) \leftrightarrow$ **REPL-PLTL** 66
$\mathbf{X}(((q \wedge \top) \vee (q \wedge \bot) \vee (q\mathbf{V}p \wedge \top)) \wedge ((q \wedge \top) \vee (q \wedge \bot) \vee (p \wedge \top)))$
68. $q\mathbf{V}p \to$ **IFF-E** 67, **HS** 61
$\mathbf{X}((q \vee (q \wedge \bot) \vee (q\mathbf{V}p \wedge \top)) \wedge (q \vee (q \wedge \bot) \vee (p \wedge \top)))$
69. $q \vee (q \wedge \bot) \to q$ **OR-L** 17, 62
70. $q \to q \vee (q \wedge \bot)$ **OR-R** 17
71. $q \leftrightarrow q \vee (q \wedge \bot)$ **IFF-RL** 69, 70
72. $\mathbf{X}((q \vee (q\mathbf{V}p \wedge \top)) \wedge (q \vee (p \wedge \top))) \leftrightarrow$ **REPL-PLTL** 71
$\mathbf{X}((q \vee (q \wedge \bot) \vee (q\mathbf{V}p \wedge \top)) \wedge (q \vee (q \wedge \bot) \vee (p \wedge \top)))$
73. $q\mathbf{V}p \to \mathbf{X}((q \vee (q\mathbf{V}p \wedge \top)) \wedge (q \vee (p \wedge \top)))$ **IFF-E** 72, **HS** 68
74. $p \to (p \to p)$ **A1-I**
75. $p \to \top$ **D1-**$\top$ 74
76. $p \to p \wedge \top$ **AND-R** 1, 75
77. $p \wedge \top \to p$ **AND-L** 1
78. $p \leftrightarrow p \wedge \top$ **IFF-RL** 76, 77
79. $\mathbf{X}((q \vee (q\mathbf{V}p \wedge \top)) \wedge (q \vee p)) \leftrightarrow$ **REPL-PLTL** 78
$\mathbf{X}((q \vee (q\mathbf{V}p \wedge \top)) \wedge (q \vee (p \wedge \top)))$
80. $q\mathbf{V}p \to \mathbf{X}((q \vee (q\mathbf{V}p \wedge \top)) \wedge (q \vee p))$ **IFF-E** 79, **HS** 73
81. $q\mathbf{V}p \to (q\mathbf{V}p \to q\mathbf{V}p)$ **A1-I**
82. $q\mathbf{V}p \to \top$ **D1-**$\top$ 81
83. $q\mathbf{V}p \to q\mathbf{V}p \wedge \top$ **AND-R** 44, 82
84. $q\mathbf{V}p \wedge \top \to q\mathbf{V}p$ **AND-L** 44
85. $q\mathbf{V}p \wedge \top \to q\mathbf{V}p$ **IFF-RL** 83, 84
86. $\mathbf{X}((q \vee (q\mathbf{V}p \wedge \top)) \wedge (q \vee p)) \leftrightarrow \mathbf{X}((q \vee q\mathbf{V}p) \wedge (q \vee p))$ **REPL-PLTL** 85
87. $q\mathbf{V}p \to \mathbf{X}((q \vee q\mathbf{V}p) \wedge (q \vee p))$ **IFF-E** 86, **HS** 80
88. $(q \vee (p \wedge q\mathbf{V}p)) \leftrightarrow (q \vee q\mathbf{V}p) \wedge (q \vee p))$ **DIST-OA**
89. $\mathbf{X}(q \vee (p \wedge q\mathbf{V}p)) \leftrightarrow \mathbf{X}((q \vee q\mathbf{V}p) \wedge (q \vee p))$ **REPL-PLTL** 88
90. $q\mathbf{V}p \to \mathbf{X}(q \vee (p \wedge q\mathbf{V}p))$ **IFF-E** 89, **HS** 87
91. $q\mathbf{V}p \leftrightarrow \mathbf{X}(q \vee (p \wedge q\mathbf{V}p))$ **IFF-RL** 16, 90

(FIX-U) $\vdash^{PLTL}_{\Delta}$ $p\mathbf{U}q \leftrightarrow q \vee (p \wedge \mathbf{X}(p\mathbf{U}q))$ (fixed point of $\mathbf{U}$)

Proof:

1. $p \to p$ **REFL**
2. $p \wedge \mathbf{X}(q \vee (p \wedge q\mathbf{V}p)) \to q\mathbf{V}p$ **FIX-V**, **AND-L**
3. $p \wedge \mathbf{X}(q \vee (p \wedge q\mathbf{V}p)) \to p$ **AND-L** 1
4. $p \wedge \mathbf{X}(q \vee (p \wedge q\mathbf{V}p)) \to p \wedge q\mathbf{V}p$ **AND-R** 2, 3
5. $p \wedge \mathbf{X}(q \vee (p \wedge q\mathbf{V}p)) \to q \vee (p \wedge q\mathbf{V}p)$ **OR-R** 4
6. $q \to q$ **REFL**
7. $q \to q \vee (p \wedge q\mathbf{V}p)$ **OR-R** 6
8. $q \vee (p \wedge \mathbf{X}(q \vee (p \wedge q\mathbf{V}p))) \to q \vee (p \wedge q\mathbf{V}p)$ **OR-L** 5, 7
9. $q \vee (p \wedge \mathbf{X}(q \vee (p \wedge q\mathbf{V}p))) \to q \vee (p \wedge q\mathbf{V}p)$ **D6-X**
10. $p \wedge q\mathbf{V}p \to \mathbf{X}(q \vee (p \wedge q\mathbf{V}p))$ **FIX-V**, **AND-L**

11. $p \wedge q\mathbf{V}p \to p$ **AND-L** 1
12. $p \wedge q\mathbf{V}p \to p \wedge \mathbf{X}(q \vee (p \wedge q\mathbf{V}p))$ **AND-R** 10, 11
13. $p \wedge q\mathbf{V}p \to q \vee (p \wedge \mathbf{X}(q \vee (p \wedge q\mathbf{V}p)))$ **OR-R** 12
14. $q \to q \vee (p \wedge \mathbf{X}(q \vee (p \wedge q\mathbf{V}p)))$ **OR-R** 6
15. $q \vee (p \wedge q\mathbf{V}p) \to q \vee (p \wedge \mathbf{X}(q \vee (p \wedge q\mathbf{V}p)))$ **OR-L** 13, 14
16. $q \vee (p \wedge q\mathbf{V}p) \leftrightarrow q \vee (p \wedge \mathbf{X}(q \vee (p \wedge q\mathbf{V}p)))$ **IFF-RL** 5, 15
17. $p\mathbf{U}q \leftrightarrow q \vee (p \wedge \mathbf{X}(p\mathbf{U}q))$ **D7-U** 16

(FIX-F) $\vdash^{PLTL}_{\Delta}$ $\mathbf{F}p \leftrightarrow p \vee \mathbf{XF}p$ (fixed point of $\mathbf{F}$)

Proof:

1. $\mathbf{F}p \leftrightarrow p \vee \top \wedge \mathbf{XF}p$ **FIX-U**, **D8-F**
2. $\mathbf{XF}p \to \top \wedge \mathbf{XF}p$ **A1-I**, **D1-**$\top$, **REFL**, **AND-R**
3. $\top \wedge \mathbf{XF}p \to \mathbf{XF}p$ **REFL**, **AND-L**
4. $\mathbf{F}p \to p \vee \mathbf{XF}p$ **IFF-E** 1, **REFL**, **OR-R**, **OR-L** 3, **HS**
5. $p \vee \mathbf{XF}p \to \mathbf{F}p$ **IFF-E** 1, **REFL**, **OR-R**, **OR-L** 2, **HS**
6. $\mathbf{F}p \leftrightarrow p \vee \mathbf{XF}p$ **IFF-RL** 4, 5

(FIX-G) $\vdash^{PLTL}_{\Delta}$ $\mathbf{G}p \leftrightarrow p \wedge \mathbf{XG}p$ (fixed point of $\mathbf{G}$)

Proof:

1. $\neg\mathbf{XF}(\neg p) \leftrightarrow \mathbf{X}(\neg\mathbf{F}(\neg p))$ **FUN-X**
2. $p \wedge \neg\mathbf{XF}(\neg p) \to \mathbf{X}(\neg\mathbf{F}(\neg p))$ **IFF-E** 1, **AND-L**
3. $p \wedge \neg\mathbf{XF}(\neg p) \to p \wedge \mathbf{X}(\neg\mathbf{F}(\neg p))$ **REFL**, **AND-L**, **AND-R** 2
4. $p \wedge \mathbf{X}(\neg\mathbf{F}(\neg p)) \to \neg\mathbf{XF}(\neg p)$ **IFF-E** 1, **AND-L**
5. $p \wedge \mathbf{X}(\neg\mathbf{F}(\neg p)) \to p \wedge \neg\mathbf{XF}(\neg p)$ **REFL**, **AND-L**, **AND-R** 4
6. $\mathbf{F}(\neg p) \leftrightarrow \neg p \vee \mathbf{XF}(\neg p)$ **FIX-F**
7. $\neg\mathbf{F}(\neg p) \to \neg(\neg p \vee \mathbf{XF}(\neg p))$ **IFF-E** 6, **CONP**, **R1-MP**
8. $\neg\mathbf{F}(\neg p) \to \neg\neg p \wedge \neg\mathbf{XF}(\neg p))$ **DM**, **IFF-E**, **HS** 7
9. $\neg\neg p \leftrightarrow p$ **DOUB**, **A3-N**, **R1-MP**, **IFF-RL**
10. $\neg\mathbf{F}(\neg p) \to p \wedge \mathbf{X}(\neg\mathbf{F}(\neg p)))$ **REPL-PLTL** 9, **IFF-E**, **R1-MP** 8, **HS** 3
11. $\neg(\neg p \vee \mathbf{XF}(\neg p)) \to \neg\mathbf{F}(\neg p)$ **IFF-E** 6, **CONP**, **R1-MP**
12. $\neg\neg p \wedge \neg\mathbf{XF}(\neg p) \to \neg\mathbf{F}(\neg p)$ **DM**, **IFF-E**, **HS** 11
13. $p \wedge \mathbf{X}(\neg\mathbf{F}(\neg p))) \to \neg\mathbf{F}(\neg p)$ **REPL-PLTL** 9, **IFF-E**, **R1-MP** 12, **HS** 5
14. $\mathbf{G}p \leftrightarrow p \wedge \mathbf{XG}p$ **IFF-RL** 10, 13; **D9-G**

(COM-GX) $\vdash^{PLTL}_{\Delta}$ $\mathbf{GX}p \leftrightarrow \mathbf{XG}p$ (commutativity of $\mathbf{G}$ and $\mathbf{X}$)

Proof:

1. $\mathbf{GX}p \leftrightarrow \mathbf{X}p \wedge \mathbf{XGX}p$ **FIX-G**
2. $\mathbf{GX}p \to \mathbf{X}(p \wedge \mathbf{GX}p)$ **IFF-E** 1, **DIST-ANDX**, **IFF-E**, **HS**
3. $p \wedge \mathbf{XG}p \to \mathbf{G}p$ **FIX-G**, **IFF-E**
4. $\mathbf{X}(p \wedge \mathbf{XG}p) \to \mathbf{XG}p$ **R2-G** 3, **MON-GX**, **R1-MP**
5. $\mathbf{GX}p \to \mathbf{XG}p$ **HS** 2, 4
6. $\mathbf{X}(p \wedge \mathbf{GX}p) \to \mathbf{GX}p$ **IFF-E** 1, **DIST-ANDX**, **IFF-E**, **HS**
7. $\mathbf{G}p \to p \wedge \mathbf{XG}p$ **FIX-G**, **IFF-E**
8. $\mathbf{XG}p \to \mathbf{X}(p \wedge \mathbf{XG}p)$ **R2-G** 7, **MON-GX**, **R1-MP**
9. $\mathbf{GX}p \leftrightarrow \mathbf{XG}p$ **HS** 8, 6; **IFF-RL** 5

(COM-FX) $\vdash_{\Delta}^{PLTL}$ $\mathbf{FX}p \leftrightarrow \mathbf{XF}p$ (commutativity of **F** and **X**)

Proof:

1.	$\mathbf{GX}(\neg p) \leftrightarrow \mathbf{XG}(\neg p)$	**COM-GX**
2.	$\mathbf{X}(\neg\mathbf{F}(\neg\neg p)) \to \neg\mathbf{F}(\neg\mathbf{X}(\neg p))$	**IFF-E** 1, **D9-G**
3.	$\mathbf{F}(\neg\mathbf{X}(\neg p)) \to \mathbf{XF}(\neg\neg p)$	**FUN-X**, **HS** 2, **INVE**, **R1-MP**
4.	$\mathbf{FX}(\neg\neg p) \to \mathbf{F}(\neg\mathbf{X}(\neg p))$	**FUN-X**, **IFF-E**, **R2-G**, **MON-GF**, **R1-MP**
5.	$\mathbf{FX}(\neg\neg p) \to \mathbf{XF}(\neg\neg p)$	**HS** 3, 4
6.	$\neg\mathbf{F}(\neg\mathbf{X}(\neg p)) \to \mathbf{X}(\neg\mathbf{F}(\neg\neg p))$	**IFF-E** 1, **D9-G**
7.	$\mathbf{XF}(\neg\neg p) \to \mathbf{F}(\neg\mathbf{X}(\neg p))$	**FUN-X**, **HS** 6, **INVE**, **R1-MP**
8.	$\mathbf{F}(\neg\mathbf{X}(\neg p)) \to \mathbf{FX}(\neg\neg p)$	**FUN-X**, **IFF-E**, **R2-G**, **MON-GF**, **R1-MP**
9.	$\mathbf{XF}(\neg\neg p) \leftrightarrow \mathbf{FX}(\neg\neg p)$	**HS** 3, 4; **IFF-RL** 5
10.	$p \leftrightarrow \neg\neg p$	**DOUB**, **A3-N**, **R1-MP**, **IFF-RL**
11.	$\mathbf{FX}p \leftrightarrow \mathbf{FX}p$	**REPL-PLTL** 10, **IFF-E**, **R1-MP** 9

(RPL-UF) $\vdash_{\Delta}^{PLTL}$ $p\mathbf{U}q \to \mathbf{F}q$

Proof:

1.	$p \to (p \to p)$	**A1-I**
2.	$p \to \top$	**D1-**$\top$ 1
3.	$\mathbf{G}(p \to \top)$	**R2-G** 2
4.	$\mathbf{G}(p \to \top) \to (q\mathbf{V}p \to q\mathbf{V}\top)$	**A5-GV**
5.	$q\mathbf{V}p \to q\mathbf{V}\top$	**R1-MP** 3, 4
6.	$p \wedge q\mathbf{V}p \to q\mathbf{V}\top$	**AND-L** 5
7.	$p \wedge q\mathbf{V}p \to q \vee q\mathbf{V}\top$	**OR-R** 6
8.	$q \to q$	**REFL**
9.	$q \to q \vee q\mathbf{V}\top$	**OR-R** 8
10.	$q \vee (p \wedge q\mathbf{V}p) \to q \vee q\mathbf{V}\top$	**OR-L** 7, 9
11.	$p\mathbf{U}q \to \mathbf{F}q$	**D7-U**, **D8-F** 10

(MON-GF) $\vdash_{\Delta}^{PLTL}$ $\mathbf{G}(p \to q) \to (\mathbf{F}p \to \mathbf{F}q)$

Proof:

1.	$(p \to q) \to (\neg q \to \neg p)$	**CONP**
2.	$\mathbf{G}((p \to q) \to (\neg q \to \neg p))$	**R2-G** 1
3.	$\mathbf{G}((p \to q) \to (\neg q \to \neg p)) \to (\mathbf{G}(p \to q) \to \mathbf{G}(\neg q \to \neg p))$	**MON-G**
4.	$\mathbf{G}(p \to q) \to \mathbf{G}(\neg q \to \neg p)$	**R1-MP** 2, 3
5.	$\mathbf{G}(\neg q \to \neg p) \to (\mathbf{G}(\neg q) \to \mathbf{G}(\neg p))$	**MON-G**
6.	$\mathbf{G}(p \to q) \to (\mathbf{G}(\neg q) \to \mathbf{G}(\neg p))$	**HS** 4, 5
7.	$(\mathbf{G}(\neg q) \to \mathbf{G}(\neg p)) \to (\neg\mathbf{G}(\neg p) \to \neg\mathbf{G}(\neg q))$	**CONP**
8.	$\mathbf{G}(p \to q) \to (\mathbf{F}p \to \mathbf{F}q)$	**HS** 6, 7; **D8-F**

(DIST-ORF) $\vdash_{\Delta}^{PLTL}$ $\mathbf{F}(p \vee q) \leftrightarrow \mathbf{F}p \vee \mathbf{F}q$ (distribution of **F** over $\vee$)

Proof:

1.	$(p \vee q)\mathbf{V}\top \leftrightarrow p\mathbf{V}\top \vee q\mathbf{V}\top$	**DIST-ORV**
2.	$(p \vee q)\mathbf{V}\top \to p\mathbf{V}\top \vee q\mathbf{V}\top$	**IFF-E** 1
3.	$(p \vee q)\mathbf{V}\top \to p \vee p\mathbf{V}\top \vee q \vee q\mathbf{V}\top$	**OR-R** 2
4.	$p \vee q \to p \vee q$	**REFL**
5.	$p \vee q \to p \vee p\mathbf{V}\top \vee q \vee q\mathbf{V}\top$	**OR-R** 4

6. $(p \vee q) \vee (p \vee q)\mathbf{V}\top \to p \vee p\mathbf{V}\top \vee q \vee q\mathbf{V}\top$ **OR-L** 3, 5
7. $p\mathbf{V}\top \vee q\mathbf{V}\top \to (p \vee q)\mathbf{V}\top$ **IFF-E** 1
8. $p\mathbf{V}\top \vee q\mathbf{V}\top \to (p \vee q) \vee (p \vee q)\mathbf{V}\top$ **OR-R** 7
9. $p \vee q \to (p \vee q) \vee (p \vee q)\mathbf{V}\top$ **OR-R** 4
10. $p \vee p\mathbf{V}\top \vee q \vee q\mathbf{V}\top \to (p \vee q) \vee (p \vee q)\mathbf{V}\top$ **OR-L** 8, 9
11. $\mathbf{F}(p \vee q) \leftrightarrow \mathbf{F}p \vee \mathbf{F}q$ **IFF-RL** 6, 10; **D8-F**

(DIST-ANDG) $\vdash_{\Delta}^{PLTL}$ $\mathbf{G}(p \wedge q) \leftrightarrow \mathbf{G}p \wedge \mathbf{G}q$ (distribution of $\mathbf{G}$ over $\wedge$)

Proof:

1. $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$ **DM**
2. $\mathbf{F}(\neg(p \wedge q)) \leftrightarrow \mathbf{F}(\neg p \vee \neg q)$ **REPL-PLTL** 1
3. $\mathbf{F}(\neg(p \wedge q)) \to \mathbf{F}(\neg p \vee \neg q)$ **IFF-E** 2
4. $\neg\mathbf{F}(\neg p \vee \neg q) \to \neg\mathbf{F}(\neg(p \wedge q))$ **CONP**, **R1-MP** 3
5. $\mathbf{F}(\neg p \vee \neg q) \to \mathbf{F}(\neg p) \vee \mathbf{F}(\neg q)$ **DIST-ORF**, **IFF-E**
6. $\neg(\mathbf{F}(\neg p) \vee \mathbf{F}(\neg q)) \to \neg\mathbf{F}(\neg p \vee \neg q)$ **CONP**, **R1-MP** 5
7. $\neg\mathbf{F}(\neg p) \wedge \neg\mathbf{F}(\neg q) \leftrightarrow \neg(\mathbf{F}(\neg p) \vee \mathbf{F}(\neg q))$ **DM**
8. $\neg\mathbf{F}(\neg p) \wedge \neg\mathbf{F}(\neg q) \to \neg(\mathbf{F}(\neg p) \vee \mathbf{F}(\neg q))$ **IFF-E** 7
9. $\neg\mathbf{F}(\neg p) \wedge \neg\mathbf{F}(\neg q) \to \neg\mathbf{F}(\neg p \vee \neg q)$ **HS** 8, 6
10. $\neg\mathbf{F}(\neg p) \wedge \neg\mathbf{F}(\neg q) \to \neg\mathbf{F}(\neg(p \wedge q))$ **HS** 9, 4
11. $\mathbf{F}(\neg p \vee \neg q) \to \mathbf{F}(\neg(p \wedge q))$ **IFF-E** 2
12. $\mathbf{F}(\neg p) \vee \mathbf{F}(\neg q) \to \mathbf{F}(\neg p \vee \neg q)$ **DIST-ORF**, **IFF-E**
13. $\mathbf{F}(\neg p) \vee \mathbf{F}(\neg q) \to \mathbf{F}(\neg(p \wedge q))$ **HS** 12, 11
14. $\neg\mathbf{F}(\neg(p \wedge q)) \to \neg(\mathbf{F}(\neg p) \vee \mathbf{F}(\neg q))$ **CONP**, **R1-MP** 13
15. $\neg(\mathbf{F}(\neg p) \vee \mathbf{F}(\neg q)) \to \neg\mathbf{F}(\neg p) \wedge \neg\mathbf{F}(\neg q)$ **IFF-E** 7
16. $\mathbf{G}(p \wedge q) \leftrightarrow \mathbf{G}p \wedge \mathbf{G}q$ **IFF-RL** 10, 15; **D9-G**

(DIST-ANDF) $\vdash_{\Delta}^{PLTL}$ $\mathbf{F}(p \wedge q) \to \mathbf{F}p \wedge \mathbf{F}q$ (distribution of $\mathbf{F}$ over $\wedge$)

Proof:

1. $p \wedge q \to p$ **REFL**, **AND-L**
2. $p \wedge q \to q$ **REFL**, **AND-L**
3. $(p \wedge q)\mathbf{V}\top \to p\mathbf{V}\top$ **R2-G** 1, **A4-GV**, **R1-MP**
4. $(p \wedge q)\mathbf{V}\top \to q\mathbf{V}\top$ **R2-G** 2, **A4-GV**, **R1-MP**
5. $(p \wedge q)\mathbf{V}\top \to p \vee p\mathbf{V}\top$ **OR-R** 3
6. $(p \wedge q)\mathbf{V}\top \to q \vee q\mathbf{V}\top$ **OR-R** 4
7. $(p \wedge q)\mathbf{V}\top \to (p \vee p\mathbf{V}\top) \wedge (q \vee q\mathbf{V}\top)$ **AND-R** 5, 6
8. $p \wedge q \to p \vee p\mathbf{V}\top$ **REFL**, **OR-R**, **AND-L**
9. $p \wedge q \to q \vee q\mathbf{V}\top$ **REFL**, **OR-R**, **AND-L**
10. $p \wedge q \to (p \vee p\mathbf{V}\top) \wedge (q \vee q\mathbf{V}\top)$ **AND-R** 8, 9
11. $\mathbf{F}(p \wedge q) \to \mathbf{F}p \wedge \mathbf{F}q$ **OR-L** 7, 10; **D8-F**

(DIST-ORG) $\vdash_{\Delta}^{PLTL}$ $\mathbf{G}p \vee \mathbf{G}q \to \mathbf{G}(p \vee q)$ (distribution of $\mathbf{G}$ over $\vee$)

Proof:

1. $\mathbf{F}(\neg p \wedge \neg q) \to \mathbf{F}(\neg p) \wedge \mathbf{F}(\neg q)$ **DIST-ANDF**
2. $\neg(p \vee q) \to (\neg p \wedge \neg q)$ **DM**, **IFF-E**
3. $\mathbf{F}(\neg(p \vee q)) \to \mathbf{F}(\neg p \wedge \neg q)$ **R2-G** 2, **MON-GF**, **R1-MP**
4. $\mathbf{F}(\neg(p \vee q)) \to \mathbf{F}(\neg p) \wedge \mathbf{F}(\neg q)$ **HS** 3, 1

5. $\neg(\mathbf{F}(\neg p) \wedge \mathbf{F}(\neg q)) \rightarrow \neg\mathbf{F}(\neg(p \vee q))$ **CONP**, **R1-MP** 4
6. $\neg\mathbf{F}(\neg p) \vee \neg\mathbf{F}(\neg q) \rightarrow \neg(\mathbf{F}(\neg p) \wedge \mathbf{F}(\neg q))$ **DM**, **IFF-E**
7. $\mathbf{G}p \vee \mathbf{G}q \rightarrow \mathbf{G}(p \vee q)$ **HS** 6, 5; **D9-G**

(LIN-FX) $\vdash_{\Delta}^{PLTL}$ $\mathbf{F}p \wedge \mathbf{F}q \rightarrow \mathbf{F}(p \wedge q) \vee \mathbf{F}(p \wedge \mathbf{XF}q) \vee \mathbf{F}(q \wedge \mathbf{XF}p)$

We just sketch here the long proof by contradiction. A formal proof may be developed along the same lines. Assume that the following properties are the case:

$$p\mathbf{V}\top \wedge q\mathbf{V}\top \quad \text{(III.2.1)}$$

$$\neg(p \wedge q)\mathbf{V}\top \wedge \neg(p \wedge \neg q \wedge q\mathbf{V}\top)\mathbf{V}\top \wedge \neg(q \wedge \neg p \wedge p\mathbf{V}\top)\mathbf{V}\top \quad \text{(III.2.2)}$$

Since (I.2.1) implies $p\mathbf{V}(p \vee \neg p) \wedge q\mathbf{V}(q \vee \neg q)$, we obtain the following sentence from **A8-V** and **R1-MP**:

$$(p \wedge q)\mathbf{V}\top \vee (p \wedge \neg q)\mathbf{V}\top \vee (q \wedge \neg p)\mathbf{V}\top \quad \text{(III.2.3)}$$

We can also derive the following sentences from the classical tautology $\neg p \wedge \neg q \rightarrow \top$ and **A4-GV**:

$$(p \wedge q)\mathbf{V}(\neg p \wedge \neg q) \rightarrow (p \wedge q)\mathbf{V}\top \quad \text{(III.2.4)}$$

$$(p \wedge \neg q \wedge q\mathbf{V}\top)\mathbf{V}(\neg p \wedge \neg q) \rightarrow (p \wedge \neg q \wedge q\mathbf{V}\top)\mathbf{V}\top \quad \text{(III.2.5)}$$

$$(q \wedge \neg p \wedge p\mathbf{V}\top)\mathbf{V}(\neg p \wedge \neg q) \rightarrow (q \wedge \neg p \wedge p\mathbf{V}\top)\mathbf{V}\top \quad \text{(III.2.6)}$$

Now, from the logical axiom scheme defining a linear time dimension (**A8-V**), we obtain the following set of mutually exclusive sentences with the help of **A5-GV**, **A11-X** and **NEG-V**$\top$:

$$(p \wedge q)\mathbf{V}(\neg p \wedge \neg q) \quad \text{(III.2.7)}$$

$$(p \wedge \neg q \wedge q\mathbf{V}\top)\mathbf{V}(\neg p \wedge \neg q) \quad \text{(III.2.8)}$$

$$(q \wedge \neg p \wedge p\mathbf{V}\top)\mathbf{V}(\neg p \wedge \neg q) \quad \text{(III.2.9)}$$

Clearly, some of these senteces must be the case. To see this, derive from (I.2.4), (I.2.5) and (I.2.6) another implication having the disjunction of their antecedant formulas as its own antencedant and similarly for the consequent. None of the three last sentences listed above were the case, the resulting implication would be contradicted with the help of (I.2.3).

The implication (I.2.4) is contradicted by (I.2.7) and the first conjunct of (I.2.2). Much the same happens to (I.2.5) considering (I.2.8) and the second conjunct therein, and to (I.2.6) taking (I.2.9) and the third conjunct of (I.2.2) into acocunt. Therefore, we reach a contradiction in any case.

The proof is completed by forming an implication with (I.2.1) and (I.2.2), removing the unnecessary conjuncts from the context of each **V** with the aid of **A4-GV** and introducing additional disjuncts to define the required **F** occurrences based on **FIX-V**. These logical operations yield the sentence in the statement of the theorem.

(LIN-G) $\vdash_{\Delta}^{PLTL}$ $\mathbf{G}(\mathbf{G}p \to q) \vee \mathbf{G}(\mathbf{G}q \to p)$

Proof:

1.	$\neg(\mathbf{G}(\mathbf{G}p \to q) \vee \mathbf{G}(\mathbf{G}q \to p)) \to$ $(\neg\mathbf{G}(\mathbf{G}p \to q) \vee \neg\mathbf{G}(\mathbf{G}q \to p))$	**DM**
2.	$\neg\mathbf{G}(\mathbf{G}p \to q) \wedge \neg\mathbf{G}(\mathbf{G}q \to p) \to$ $\mathbf{F}(\neg(\mathbf{G}p \to q))$	**DUAL-GF**, **IFF-E**, **AND-L**
3.	$(\mathbf{G}p \to \neg\neg q) \to (\mathbf{G}p \to q)$	**DOUB**, **LTRAN**, **R1-MP**
4.	$\neg(\mathbf{G}p \to q) \to \mathbf{G}p \wedge \neg q$	**INVE**, **R1-MP** 3, **D4-AND**
5.	$\mathbf{F}(\neg(\mathbf{G}p \to q)) \to \mathbf{F}(\mathbf{G}p \wedge \neg q)$	**R2-G** 4, **MON-GF**, **R1-MP**
6.	$\neg\mathbf{G}(\mathbf{G}p \to q) \wedge \neg\mathbf{G}(\mathbf{G}q \to p) \to$ $\mathbf{F}(\neg(\mathbf{G}q \to p))$	**DUAL-GF**, **IFF-E**, **AND-L**
7.	$(\mathbf{G}q \to \neg\neg p) \to (\mathbf{G}q \to p)$	**DOUB**, **LTRAN**, **R1-MP**
8.	$\neg(\mathbf{G}q \to p) \to \mathbf{G}q \wedge \neg p$	**INVE**, **R1-MP** 7, **D4-AND**
9.	$\mathbf{F}(\neg(\mathbf{G}q \to p)) \to \mathbf{F}(\mathbf{G}q \wedge \neg p)$	**R2-G** 8, **MON-GF**, **R1-MP**
10.	$\neg(\mathbf{G}(\mathbf{G}p \to q) \vee \mathbf{G}(\mathbf{G}q \to p)) \to$ $\mathbf{F}(\mathbf{G}p \wedge \neg q) \wedge \mathbf{F}(\mathbf{G}q \wedge \neg p)$	**HS** 2, 5; **HS** 6, 9; **AND-R**, **HS** 1
11.	$\mathbf{F}(\mathbf{G}p \wedge \neg q) \wedge \mathbf{F}(\mathbf{G}q \wedge \neg p) \to$ $\mathbf{F}(\mathbf{G}p \wedge \neg q \wedge \mathbf{G}q \wedge \neg p) \vee$ $\mathbf{F}(\mathbf{G}p \wedge \neg q \wedge \mathbf{X}(\mathbf{G}q \wedge \neg p)) \vee$ $\mathbf{F}(\mathbf{G}q \wedge \neg p \wedge \mathbf{X}(\mathbf{G}p \wedge \neg q))$	**LIN-FX**
12.	$p \to \neg\neg p$	**DOUB**, **A3-N**, **R1-MP**
13.	$\neg(p \to \neg\neg p) \to \neg(p \to p)$	**LTRAN**, **R1-MP** 12, **INVE**, **R1-MP**
14.	$p \wedge \neg p \to \bot$	**D4-AND** 13, **D2-BOT**
15.	$\mathbf{G}p \wedge \neg q \wedge \mathbf{G}q \wedge \neg p \to p$	**REFL-G**, **AND-L**
16.	$\mathbf{G}p \wedge \neg q \wedge \mathbf{G}q \wedge \neg p \to p \wedge \neg p$	**REFL**, **AND-L**, **AND-R** 15
17.	$\mathbf{F}(\mathbf{G}p \wedge \neg q \wedge \mathbf{G}q \wedge \neg p) \to \mathbf{F}\bot$	**HS** 14, 16; **R2-G**, **MON-GF**, **R1-MP**
18.	$\mathbf{G}p \neg q \wedge \mathbf{X}(\mathbf{G}q \wedge \neg p) \to \mathbf{X}p$	**REPL-GX**, **AND-L**
19.	$\mathbf{X}(\mathbf{G}q \wedge \neg p) \to \mathbf{X}\mathbf{G}q \wedge \mathbf{X}(\neg p)$	**DIST-ANDX**, **IFF-E**
20.	$\mathbf{X}(\mathbf{G}q \wedge \neg p) \to \mathbf{X}(\neg p)$	**REFL**, **AND-L**, **HS** 19
21.	$\mathbf{G}p \wedge \neg q \wedge \mathbf{X}(\mathbf{G}q \wedge \neg p) \to \mathbf{X}p \wedge \mathbf{X}(\neg p)$	**AND-L** 20, **AND-R** 18
22.	$\mathbf{X}p \wedge \mathbf{X}(\neg p) \to \mathbf{X}(p \wedge \neg p)$	**DIST-ANDX**, **IFF-E**
23.	$\mathbf{X}p \wedge \mathbf{X}(\neg p) \to \mathbf{X}\bot$	**R2-G** 14, **MON-GX**, **R1-MP**, **HS** 22
24.	$\mathbf{X}p \wedge \mathbf{X}(\neg p) \to \neg\mathbf{X}\top$	**D2-BOT** 23, **FUN-X**, **IFF-E**, **HS**
25.	$\mathbf{X}p \wedge \mathbf{X}(\neg p) \to \bot$	**NEG-L**, **A11-X**, **R1-MP**, **HS** 24
26.	$\mathbf{F}(\mathbf{G}p \wedge \neg q \wedge \mathbf{X}(\mathbf{G}q \wedge \neg p)) \to \mathbf{F}\bot$	**HS** 21, 25; **R2-G**, **MON-GF**, **R1-MP**
27.	$q \to \neg\neg q$	**DOUB**, **A3-N**, **R1-MP**
28.	$\neg(q \to \neg\neg q) \to \neg(q \to q)$	**LTRAN**, **R1-MP** 27, **INVE**, **R1-MP**
29.	$q \wedge \neg q \to \bot$	**D4-AND** 27, **D2-BOT**
30.	$\mathbf{G}((\mathbf{G}p \wedge \neg q) \to p)$	**REFL-G**, **AND-L**, **R2-G**

31. $\mathbf{G}q \wedge \neg p \wedge \mathbf{X}(\mathbf{G}p \wedge \neg q) \rightarrow \mathbf{X}p$ **MON-GX**, **R1-MP** 30, **AND-L**
32. $\mathbf{G}q \wedge \neg p \wedge \mathbf{X}(\mathbf{G}p \wedge \neg q) \rightarrow \mathbf{X}q \wedge \mathbf{X}(\neg q)$ **REPL-GX**, **AND-L**, **AND-R** 31
33. $\mathbf{X}(q) \wedge \mathbf{X}(\neg q) \rightarrow \mathbf{X}(q \wedge \neg q)$ **DIST-ANDX**, **IFF-E**
34. $\mathbf{X}q \wedge \mathbf{X}(\neg q) \rightarrow \mathbf{X}\bot$ **R2-G** 29, **MON-GX**, **R1-MP**, **HS** 33
35. $\mathbf{X}q \wedge \mathbf{X}(\neg q) \rightarrow \neg\mathbf{X}\top$ **D2-BOT** 31, **FUN-X**, **IFF-E**, **HS**
36. $\mathbf{X}q \wedge \mathbf{X}(\neg q) \rightarrow \bot$ **NEG-L**, **A11-X**, **R1-MP**, **HS** 35
37. $\mathbf{F}(\mathbf{G}q \wedge \neg p \wedge \mathbf{X}(\mathbf{G}p \wedge \neg q)) \rightarrow \mathbf{F}\bot$ **HS** 32, 36; **R2-G**, **MON-GF**, **R1-MP**
38. $\mathbf{F}(\mathbf{G}p \wedge \neg q) \wedge \mathbf{F}(\mathbf{G}q \wedge \neg p) \rightarrow \mathbf{F}\bot$ **OR-L** 17, 26, 37; **HS** 11
39. $\mathbf{F}(\mathbf{G}p \wedge \neg q) \wedge \mathbf{F}(\mathbf{G}q \wedge \neg p) \rightarrow \neg\mathbf{G}\top$ **D2-**$\bot$ 38, **DUAL-GF**, **IFF-E**, **HS**
40. $\mathbf{F}(\mathbf{G}p \wedge \neg q) \wedge \mathbf{F}(\mathbf{G}q \wedge \neg p) \rightarrow \neg\top$ **NEG-L**, **G**$\top$, **R1-MP**, **HS** 39, **D2-**$\top$
41. $\mathbf{G}(\mathbf{G}p \rightarrow q) \vee \mathbf{G}(\mathbf{G}q \rightarrow p)$ **HS** 10, 40; **A3-N**, **R1-MP**, **REFL**, **R1-MP**

(DIST-ORGF) $\vdash_{\Delta}^{PLTL}$ $\mathbf{GF}(p \vee q) \leftrightarrow \mathbf{GF}p \vee \mathbf{GF}q$

Proof:

1. $\mathbf{GF}p \vee \mathbf{GF}q \rightarrow \mathbf{G}(\mathbf{F}p \vee \mathbf{F}q)$ **DIST-ORG**
2. $\mathbf{F}p \vee \mathbf{F}q \rightarrow \mathbf{F}(p \vee q)$ **DIST-ORF**, **IFF-E**
3. $\mathbf{G}(\mathbf{F}p \vee \mathbf{F}q) \rightarrow \mathbf{GF}(p \vee q)$ **R2-G** 2, **MON-G**, **R1-MP**
4. $\mathbf{GF}p \vee \mathbf{GF}q \rightarrow \mathbf{GF}(p \vee q)$ **HS** 1, 3
5. $\mathbf{F}(\mathbf{F}p \vee (\mathbf{G}(\neg p) \wedge q)) \rightarrow \mathbf{FF}p \vee \mathbf{F}(\mathbf{G}(\neg p) \wedge q)$ **DIST-ORF**
6. $\mathbf{FF}p \rightarrow \mathbf{F}p \vee \mathbf{F}(\mathbf{G}(\neg p) \wedge q)$ **IDEM-F**, **IFF-E**, **OR-R**
7. $\mathbf{FF}p \vee \mathbf{F}(\mathbf{G}(\neg p) \wedge q) \rightarrow \mathbf{F}p \vee \mathbf{F}(\mathbf{G}(\neg p) \wedge q)$ **REFL**, **OR-R**, **OR-L** 6
8. $\mathbf{F}(\mathbf{F}p \vee (\mathbf{G}(\neg p) \wedge q)) \rightarrow \mathbf{F}p \vee \mathbf{F}(\mathbf{G}(\neg p) \wedge q)$ **HS** 5, 7
9. $\mathbf{GF}(\mathbf{F}p \vee \mathbf{G}(\neg p) \wedge q) \rightarrow$ $\mathbf{G}(\mathbf{F}p \vee \mathbf{F}(\mathbf{G}(\neg p) \wedge q))$ **R2-G** 8, **MON-G**, **R1-MP**
10. $p \rightarrow \mathbf{F}p$ **REFL**, **OR-R**, **D7-F**
11. $p \vee q \rightarrow \mathbf{F}p \vee q$ **REFL**, **OR-R**, **OR-L** 10
12. $p \vee q \rightarrow \mathbf{F}p \vee \mathbf{G}(\neg p)$ **DUAL-FG**, **A1-I**, **R1-MP**
13. $p \vee q \rightarrow \mathbf{F}p \vee \mathbf{G}(\neg p) \wedge q$ **AND-R** 11, 12; **DIST-IFA**, **HS**
14. $\mathbf{F}(p \vee q) \rightarrow \mathbf{F}(\mathbf{F}p \vee \mathbf{G}(\neg p) \wedge q)$ **R2-G** 13, **MON-GF**, **R1-MP**
15. $\mathbf{GF}(p \vee q) \rightarrow \mathbf{GF}(\mathbf{F}p \vee \mathbf{G}(\neg p) \wedge q)$ **R2-G** 14, **MON-G**, **R1-MP**
16. $\mathbf{G}(\mathbf{F}p \vee \mathbf{F}(\mathbf{G}(\neg p) \wedge q)) \rightarrow$ $\mathbf{GF}p \vee \mathbf{FF}(\mathbf{G}(\neg p) \wedge q)$ **MON-GF**, **D3-OR**, **D9-G**
17. $\mathbf{FF}(\mathbf{G}(\neg p) \wedge q) \rightarrow \mathbf{GF}p \vee \mathbf{F}(\mathbf{G}(\neg p) \wedge q)$ **IDEM-F**, **IFF-E**, **OR-R**
18. $\mathbf{GF}p \vee \mathbf{FF}(\mathbf{G}(\neg p) \wedge q) \rightarrow \mathbf{GF}p \vee \mathbf{F}(\mathbf{G}(\neg p) \wedge q)$ **REFL**, **OR-R**, **OR-L** 17
19. $\mathbf{G}(\mathbf{F}p \vee \mathbf{F}(\mathbf{G}(\neg p) \wedge q)) \rightarrow \mathbf{GF}p \vee \mathbf{F}(\mathbf{G}(\neg p) \wedge q)$ **HS** 16, 18
20. $\mathbf{GF}(p \vee q) \rightarrow \mathbf{GF}p \vee \mathbf{F}(\mathbf{G}(\neg p) \wedge q)$ **HS** 15, 9; **HS** 19
21. $\mathbf{F}(\mathbf{F}q \vee \mathbf{G}(\neg q) \wedge p) \rightarrow \mathbf{FF}q \vee \mathbf{F}(\mathbf{G}(\neg q) \wedge p)$ **DIST-ORF**
22. $\mathbf{FF}q \rightarrow \mathbf{F}q \vee \mathbf{F}(\mathbf{G}(\neg q) \wedge p)$ **IDEM-F**, **IFF-E**, **OR-R**
23. $\mathbf{FF}q \vee \mathbf{F}(\mathbf{G}(\neg q) \wedge p) \rightarrow \mathbf{F}q \vee \mathbf{F}(\mathbf{G}(\neg q) \wedge p)$ **REFL**, **OR-R**, **OR-L** 22
24. $\mathbf{F}(\mathbf{F}q \vee \mathbf{G}(\neg q) \wedge p) \rightarrow \mathbf{F}q \vee \mathbf{F}(\mathbf{G}(\neg q) \wedge p)$ **HS** 21, 23
25. $\mathbf{GF}(\mathbf{F}q \vee \mathbf{G}(\neg q) \wedge p) \rightarrow$ $\mathbf{G}(\mathbf{F}q \vee \mathbf{F}(\mathbf{G}(\neg q) \wedge p))$ **R2-G** 24, **MON-G**, **R1-MP**
26. $q \rightarrow \mathbf{F}q$ **REFL**, **OR-R**, **D7-F**
27. $p \vee q \rightarrow \mathbf{F}q \vee p$ **REFL**, **OR-R**, **OR-L** 26
28. $p \vee q \rightarrow \mathbf{F}q \vee \mathbf{G}(\neg q)$ **DUAL-FG**, **A1-I**, **R1-MP**
29. $p \vee q \rightarrow \mathbf{F}q \vee \mathbf{G}(\neg q) \wedge p$ **AND-R** 27, 28; **DIST-IFA**, **HS**
30. $\mathbf{F}(p \vee q) \rightarrow \mathbf{F}(\mathbf{F}q \vee \mathbf{G}(\neg q) \wedge p)$ **R2-G** 29, **MON-GF**, **R1-MP**
31. $\mathbf{GF}(p \vee q) \rightarrow \mathbf{GF}(\mathbf{F}q \vee \mathbf{G}(\neg q) \wedge p)$ **R2-G** 30, **MON-G**, **R1-MP**

32. $\mathbf{G}(\mathbf{F}q \vee \mathbf{F}(\mathbf{G}(\neg q) \wedge p)) \rightarrow$ **MON-GF**, **D3-OR**, **D9-G**
$\mathbf{GF}q \vee \mathbf{FF}(\mathbf{G}(\neg q) \wedge p)$
33. $\mathbf{FF}(\mathbf{G}(\neg q) \wedge p) \rightarrow \mathbf{GF}q \vee \mathbf{F}(\mathbf{G}(\neg q) \wedge p)$ **IDEM-F**, **IFF-E**, **OR-R**
34. $\mathbf{GF}q \vee \mathbf{FF}(\mathbf{G}(\neg q) \wedge p) \rightarrow \mathbf{GF}q \vee \mathbf{F}(\mathbf{G}(\neg q) \wedge p)$ **REFL**, **OR-R**, **OR-L** 33
35. $\mathbf{G}(\mathbf{F}q \vee \mathbf{F}(\mathbf{G}(\neg q) \wedge p)) \rightarrow \mathbf{GF}q \vee \mathbf{F}(\mathbf{G}(\neg q) \wedge p)$ **HS** 33, 35
36. $\mathbf{GF}(p \vee q) \rightarrow \mathbf{GF}q \vee \mathbf{F}(\mathbf{G}(\neg q) \wedge p)$ **HS** 30, 25; **HS** 35
37. $\mathbf{GF}(p \vee q) \rightarrow$ **AND-R** 20, 36; **DIST-AO**
$\mathbf{GF}p \vee \mathbf{GF}q \vee \mathbf{F}(\mathbf{G}(\neg p) \wedge q) \wedge \mathbf{F}(\mathbf{G}(\neg q) \wedge p)$
38. $\neg\mathbf{F}(\mathbf{G}(\neg p) \vee \neg\neg q) \vee \neg\mathbf{F}(\mathbf{G}(\neg q) \vee \neg\neg p)$ **LIN-G**, **D9-G**, **D3-OR**
39. $\neg\neg p \leftrightarrow p$ **DOUB**, **A3-N**, **R1-MP**, **IFF-RL**
40. $\neg\mathbf{F}(\mathbf{G}(\neg p) \vee \neg\neg q) \vee \neg\mathbf{F}(\mathbf{G}(\neg q) \vee p)$ **REPL-LTPL** 39, **IFF-E**, **R1-MP** 38
41. $\neg\neg q \leftrightarrow q$ **DOUB**, **A3-N**, **R1-MP**, **IFF-RL**
42. $\neg\mathbf{F}(\mathbf{G}(\neg p) \vee q) \vee \neg\mathbf{F}(\mathbf{G}(\neg q) \vee p)$ **REPL-LTPL** 41, **IFF-E**, **R1-MP** 40
43. $\neg(\mathbf{F}(\mathbf{G}(\neg p) \wedge q) \wedge \mathbf{F}(\mathbf{G}(\neg q) \wedge p))$ **D9-G**, **DM**, **IFF-E**, **R1-MP** 42, **D9-G**
44. $\mathbf{GF}(p \vee q) \rightarrow \mathbf{GF}p \vee \mathbf{GF}q$ **EXP**, **D3-OR**, **R1-MP** 43, **HS** 37
45. $\mathbf{GF}(p \vee q) \leftrightarrow \mathbf{GF}p \vee \mathbf{GF}q$ **IFF-RL** 4, 44

(DIST-ANDFG) $\vdash_{\Delta}^{PLTL}$ $\mathbf{FG}(p \wedge q) \leftrightarrow \mathbf{FG}p \wedge \mathbf{FG}q$

(COM-FG) $\vdash_{\Delta}^{PLTL}$ $\mathbf{FG}p \rightarrow \mathbf{GF}p$

Proof:

1. $p \rightarrow \mathbf{F}p$ **REFL**, **OR-R**, **D9-F**
2. $\mathbf{G}p \rightarrow \mathbf{GF}p$ **R2-G** 1, **MON-G**, **R1-MP**
3. $\mathbf{FG}p \rightarrow \mathbf{FGF}p$ **R2-G** 2, **MON-GF**, **R1-MP**
4. $\mathbf{GF}(\mathbf{F}p \vee \mathbf{G}(\neg p)) \rightarrow$ **DIST-ORGF**
$\mathbf{GFF}p \vee \mathbf{GFG}(\neg p)$
5. $\mathbf{F}p \vee \mathbf{G}(\neg p)$ **DUAL-GF**, **CONP**, **R1-MP**, **D3-OR**
6. $\mathbf{G}(\mathbf{F}p \vee \mathbf{G}(\neg p)) \rightarrow$ **REFL**, **OR-R**, **R2-G**, **MON-G**, **R1-MP**
$\mathbf{GF}(\mathbf{F}p \vee \mathbf{G}(\neg p))$
7. $\mathbf{GF}(\mathbf{F}p \vee \mathbf{G}(\neg p))$ **R2-G** 5, **R1-MP** 6
8. $\neg\mathbf{GFG}(\neg p) \rightarrow \mathbf{GFF}p$ **R1-MP** 7, 4, **D3-OR**
9. $\mathbf{GFF}p \rightarrow \mathbf{GF}p$ **IDEM-F**, **IFF-E**, **R2-G**, **MON-G**, **R1-MP**
10. $\neg\mathbf{GFG}(\neg p) \rightarrow \mathbf{GF}p$ **HS** 8, 9
11. $\mathbf{FG}(\neg p) \rightarrow \mathbf{F}(\neg\mathbf{F}p)$ **DUAL-GF**, **IFF-E**, **R2-G**, **MON-GF**, **R1-MP**
12. $\mathbf{GF}p \rightarrow \neg\mathbf{FG}(\neg p)$ **CONP**, **R1-MP** 11, **D9-G**
13. $\mathbf{FGF}p \rightarrow \mathbf{F}(\neg\mathbf{FG}(\neg p))$ **R2-G** 12, **MON-GF**, **R1-MP**
14. $\neg\neg\mathbf{FGF}p \rightarrow \neg\mathbf{GFG}(\neg p)$ **CONP**, **R1-MP** 13, **D9-G**, **CONP**, **R1-MP**
15. $\mathbf{FGF}p \rightarrow \mathbf{GF}p$ **DOUB**, **A3-N**, **R1-MP**, **HS** 14, **HS** 10
16. $\mathbf{FG}p \rightarrow \mathbf{GF}p$ **HS** 3, 15

(MON-GU) $\vdash_{\Delta}^{PLTL}$ $\mathbf{G}(p \rightarrow q) \rightarrow (p\mathbf{U}r \rightarrow q\mathbf{U}r)$
[or $\mathbf{G}(p \rightarrow q) \rightarrow (r\mathbf{U}p \rightarrow r\mathbf{U}q)$]

(MON-GW) $\vdash_{\Delta}^{PLTL}$ $\mathbf{G}(p \rightarrow q) \rightarrow (p\mathbf{W}r \rightarrow q\mathbf{W}r)$
[or $\mathbf{G}(p \rightarrow q) \rightarrow (r\mathbf{W}p \rightarrow r\mathbf{W}q)$]

(RPL-WUF) $\vdash_{\Delta}^{PLTL}$ $p\mathbf{U}q \leftrightarrow p\mathbf{W}q \wedge \mathbf{F}q$

(TRAN-W) $\{p \to q\mathbf{W}r, r \to q\mathbf{W}s\} \vdash_{\Delta}^{PLTL} p \to q\mathbf{W}s$ (transitivity of $\mathbf{W}$)

The set of axiom schemes {**DUAL-GF**, **REFL-G**, **MON-G**, **RPL-GX**, **EXP-GX**, **FUN-X**, **MON-X**, **A10-G**, **FIX-U**, **RPL-UF**} together with **R1-MP** corresponds precisely to the propositional part of the axiomatization of a temporal logic of programs proposed in (Manna and Pnueli 1983).

III.3 Propositional Branching Time Logic

Postulating the axiomatization of *branching time propositional logic* ($PBTL$) discussed in Section 2.5, the theorems over $\Delta \in \mathsf{obj}\ \mathbf{Sig}^{PBTL}$ below are provable:

(DUAL-AE) $\vdash_{\Delta}^{PBTL} \mathbf{E}(\neg p) \leftrightarrow \neg\mathbf{A}p$ (duality between $\mathbf{A}$ and $\mathbf{E}$)

(REPL-PBTL) $\{x \leftrightarrow y\} \vdash_{\Delta}^{PBTL} p[q\backslash x] \leftrightarrow p[q\backslash y]$ (replacement)

We prove this theorem by structural induction on $\mathcal{G}^{PBTL}(\Delta)$. The proof is essentially the same as for $PLTL$, with an additional inductive step. We have to examine the case where $p \equiv \mathbf{A}r$:

1.	$x \leftrightarrow y$	**Ass**
2.	$r[q\backslash x] \leftrightarrow r[q\backslash y]$	**Ind. Hyp.** 1
3.	$r[q\backslash x] \to r[q\backslash y]$	**IFF-E** 2
4.	$\mathbf{A}(r[q\backslash x] \to r[q\backslash y])$	**R4-A** 3
5.	$\mathbf{A}(r[q\backslash x] \to r[q\backslash y]) \to (\mathbf{A}(r[q\backslash x]) \to \mathbf{A}(r[q\backslash y]))$	**A13-A**
6.	$\mathbf{A}(r[q\backslash x]) \to \mathbf{A}(r[q\backslash y])$	**R1-MP** 4, 5
7.	$r[q\backslash y] \to r[q\backslash x]$	**IFF-E** 2
8.	$\mathbf{A}(r[q\backslash y] \to r[q\backslash x])$	**R4-A** 7
9.	$\mathbf{A}(r[q\backslash y] \to r[q\backslash x]) \to (\mathbf{A}(r[q\backslash y]) \to \mathbf{A}(r[q\backslash x]))$	**A13-A**
10.	$\mathbf{A}(r[q\backslash y]) \to \mathbf{A}(r[q\backslash x])$	**R1-MP** 8, 9
11.	$\mathbf{A}(r[q\backslash x]) \leftrightarrow \mathbf{A}(r[q\backslash y])$	**IFF-RL** 6, 10
12.	$(\mathbf{A}r)[q\backslash x] \leftrightarrow (\mathbf{A}r)[q\backslash y]$	**DEF** $[\cdot]$ 11

(E-R) $\vdash_{\Delta}^{PBTL} p \to \mathbf{E}p$

Proof:

1.	$\mathbf{A}(\neg p) \to \neg p$	**A14-A**
2.	$(\mathbf{A}(\neg p) \to \neg p) \to (\neg\neg p \to \neg\mathbf{A}(\neg p))$	**CONP**
3.	$\neg\neg p \to \neg\mathbf{A}(\neg p)$	**R1-MP** 1, 2
4.	$\neg\neg\neg p \to \neg p$	**DOUB**
5.	$(\neg\neg\neg p \to \neg p) \to (p \to \neg\neg p)$	**A3-N**
6.	$p \to \neg\neg p$	**R1-MP** 4, 5
7.	$p \to \mathbf{E}p$	**HS** 3, 6; **D11-E**

(MOD-B) $\vdash_{\Delta}^{PBTL} p \to \mathbf{AE}p$

Proof:

1.	$\mathbf{E}p \to \mathbf{AE}p$	**A15-EA**
2.	$p \to \mathbf{E}p$	**E-R**
3.	$p \to \mathbf{AE}p$	**HS** 2, 1

(CANC-EA) $\vdash_{\triangle}^{PBTL}$ $\mathbf{EA}p \to p$ (cancelation of **EA**)

Proof:

1.	$\neg p \to \mathbf{AE}(\neg p)$	**MON-B**
2.	$\mathbf{EA}(\neg\neg p) \to \neg\neg p$	**CONP**, **R1-MP** 1, **D11-E**
3.	$\neg\neg p \leftrightarrow p$	**DOUB**, **A3-N**, **R1-MP**, **IFF-RL**
4.	$\mathbf{EA}p \to p$	**REPL-PBTL** 3, **IFF-E**, **R1-MP** 2

(IDEM-A) $\vdash_{\triangle}^{PBTL}$ $\mathbf{A}p \to \mathbf{AA}p$[5] (idempotence of **A**)

Proof:

1.	$\mathbf{E}(\neg p) \to \mathbf{AE}(\neg p)$	**A15-EA**
2.	$(\mathbf{E}(\neg p) \to \mathbf{AE}(\neg p)) \to (\neg\mathbf{AE}(\neg p) \to \neg\mathbf{E}(\neg p))$	**CONP**
3.	$\neg\mathbf{AE}(\neg p) \to \neg\mathbf{E}(\neg p)$	**R1-MP** 1, 2
4.	$\neg\mathbf{A}(\neg\mathbf{A}(\neg\neg p)) \to \neg\neg\mathbf{A}(\neg\neg p)$	**D11-E**
5.	$\neg\neg p \leftrightarrow p$	**DOUB**, **A3-N**, **R1-MP**, **IFF-RL**
6.	$\neg\mathbf{A}(\neg\mathbf{A}p) \to \neg\mathbf{A}(\neg\mathbf{A}(\neg\neg p))$	**REPL-PBTL** 5, **IFF-E**
7.	$\neg\mathbf{A}(\neg\mathbf{A}p) \to \neg\neg\mathbf{A}(\neg\neg p)$	**HS** 6, 4
8.	$\neg\neg\mathbf{A}(\neg\neg p) \to \neg\neg\mathbf{A}p$	**REPL-PBTL** 6, **IFF-E**
9.	$\neg\mathbf{A}(\neg\mathbf{A}p) \to \neg\neg\mathbf{A}p$	**HS** 7, 8
10.	$\neg\neg\mathbf{A}p \to \mathbf{A}p$	**DOUB**
11.	$\mathbf{EA}p \to \mathbf{A}p$	**HS** 9, 10; **D11-E**
12.	$\mathbf{A}(\mathbf{EA}p \to \mathbf{A}p)$	**R4-A** 11
13.	$\mathbf{A}(\mathbf{EA}p \to \mathbf{A}p) \to (\mathbf{AEA}p \to \mathbf{AA}p)$	**A13-A**
14.	$\mathbf{AEA}p \to \mathbf{AA}p$	**R1-MP** 12, 13
15.	$\mathbf{A}p \to \mathbf{AEA}p$	**MOD-B**
16.	$\mathbf{A}p \to \mathbf{AA}p$	**HS** 14, 15

(MON-AE) $\vdash_{\triangle}^{PBTL}$ $\mathbf{A}(p \to q) \to (\mathbf{E}p \to \mathbf{E}q)$

Substitute **G** for **A** and **F** for **E** in the proof of **MON-GF** presented in Section I.2 and change the labels that justify each proof step accordingly to obtain a proof of this theorem.

(DIST-ORE) $\vdash_{\triangle}^{PBTL}$ $\mathbf{E}(p \vee q) \leftrightarrow \mathbf{E}p \vee \mathbf{E}q$ (distribution of **E** over $\vee$)

Proof:

1.	$p \to p \vee q$	**REFL**, **OR-R**
2.	$\mathbf{E}p \to \mathbf{E}(p \vee q)$	**R4-G** 1, **MON-AE**, **R1-MP**
3.	$q \to p \vee q$	**REFL**, **OR-R**
4.	$\mathbf{E}q \to \mathbf{E}(p \vee q)$	**R4-G** 3, **MON-AE**, **R1-MP**
5.	$\mathbf{E}p \vee \mathbf{E}q \to \mathbf{E}(p \vee q)$	**OR-L** 2, 4

[5] Note that the converse is directly derivable from **A14-A**.

6. $\neg p \to (\neg q \to \neg p)$ **A1-I**
7. $\neg p \to (\neg q \to \neg q)$ **REFL**, **A1-I**, **R1-MP**
8. $\neg p \to (\neg q \to \neg p \wedge \neg q)$ **AND-R** 6, 7; **DIST-IA**, **R1-MP**
9. $\mathbf{A}(\neg p) \to \mathbf{A}(\neg q \to \neg p \wedge \neg q)$ **R4-A** 8, **A13-A**, **R1-MP**
10. $\mathbf{A}(\neg p) \to (\mathbf{A}(\neg q) \to \mathbf{A}(\neg p \wedge \neg q))$ **A13-A**, **HS** 9
11. $(\mathbf{A}(\neg p) \wedge \mathbf{A}(\neg q) \to \mathbf{A}(\neg q)) \to$ $(\mathbf{A}(\neg p) \wedge \mathbf{A}(\neg q) \to \mathbf{A}(\neg p \wedge \neg q))$ **AND-L** 10, **A2-I**, **R1-MP**
12. $\mathbf{A}(\neg p) \wedge \mathbf{A}(\neg q) \to \mathbf{A}(\neg p)$ **REFL**, **AND-L**
13. $\mathbf{A}(\neg p) \wedge \mathbf{A}(\neg q) \to \mathbf{A}(\neg p \wedge \neg q)$ **R1-MP** 12, 11
14. $\neg\mathbf{A}(\neg p \wedge \neg q) \to \neg(\mathbf{A}(\neg p) \wedge \mathbf{A}(\neg q))$ **CONP**, **R1-MP** 13
15. $\mathbf{A}(\neg p \wedge \neg q) \to \mathbf{A}(\neg(p \vee q))$ **DM**, **R4-A**, **A13-A**, **R1-MP**
16. $\neg\mathbf{A}(\neg(p \vee q)) \to \neg(\mathbf{A}(\neg p) \wedge \mathbf{A}(\neg q))$ **CONP**, **R1-MP** 15, **HS** 14
17. $\mathbf{E}(p \vee q) \to \mathbf{E}p \vee \mathbf{E}q$ **DM**, **HS** 16, **D11-E**
18. $\mathbf{E}(p \vee q) \leftrightarrow \mathbf{E}p \vee \mathbf{E}q$ **IFF-RL** 5, 17

(DIST-ANDA) $\vdash^{PBTL}_{\Delta}$ $\mathbf{A}(p \wedge q) \leftrightarrow \mathbf{A}p \wedge \mathbf{A}q$ (distribution of **A** over $\wedge$)

Substitute **G** for **A** and **F** for **E** in the proof of **DIST-ANDG** presented in Section I.2 and change the labels that justify each proof step accordingly to obtain a proof of this theorem.

(DIST-ANDE) $\vdash^{PBTL}_{\Delta}$ $\mathbf{E}(p \wedge q) \to \mathbf{E}p \wedge \mathbf{E}q$ (distribution of **E** overr $\wedge$)

This proof is analogous to that of **DIST-ANDF**.

(DIST-ORA) $\vdash^{PBTL}_{\Delta}$ $\mathbf{A}p \vee \mathbf{A}q \to \mathbf{A}(p \vee q)$ (distribution of **A** over $\vee$)

This proof is analogous to that of **DIST-ORG**.

(COM-AG) $\vdash^{PBTL}_{\Delta}$ $\mathbf{AG}p \to \mathbf{GA}p$ (commutativity of **G** and **A**)

1. $(\mathbf{E}(\neg p))\mathbf{V}\top \to \mathbf{E}((\neg p)\mathbf{V}\top)$ **A23-EV**
2. $(\neg\mathbf{A}(\neg\neg p))\mathbf{V}\top \to \neg\mathbf{A}(\neg(\neg p)\mathbf{V}\top)$ **D11-E** 1
3. $\neg\neg\mathbf{A}(\neg(\neg p)\mathbf{V}\top) \to \neg(\neg\mathbf{A}(\neg\neg p))\mathbf{V}\top$ **CONP**, **R1-MP** 2
4. $\mathbf{A}(\neg(\neg p)\mathbf{V}\top) \to \neg\neg\mathbf{A}(\neg(\neg p)\mathbf{V}\top)$ **DOUB**, **A3-N**, **R1-MP**
5. $\mathbf{A}(\neg(\neg p)\mathbf{V}\top) \to \neg(\neg\mathbf{A}(\neg\neg p))\mathbf{V}\top$ **HS** 4, 3
6. $\mathbf{A}(\neg\neg p \to p)$ **DOUB**, **R4-A**
7. $\mathbf{A}(\neg\neg p \to p) \to (\mathbf{A}(\neg\neg p) \to \mathbf{A}p)$ **A13-A**
8. $\mathbf{A}(\neg\neg p) \to \mathbf{A}p$ **R1-MP** 6, 7
9. $\neg\mathbf{A}p \to \neg\mathbf{A}(\neg\neg p)$ **CONP**, **R1-MP** 8
10. $\mathbf{G}(\neg\mathbf{A}p \to \neg\mathbf{A}(\neg\neg p))$ **R2-G** 9
11. $(\neg\mathbf{A}p)\mathbf{V}\top \to (\neg\mathbf{A}(\neg\neg p))\mathbf{V}\top$ **A4-GV**, **R1-MP** 10
12. $\neg(\neg\mathbf{A}(\neg\neg p))\mathbf{V}\top \to \neg(\neg\mathbf{A}p)\mathbf{V}\top$ **CONP**, **R1-MP** 11
13. $\mathbf{A}(\neg(\neg p)\mathbf{V}\top) \to \neg(\neg\mathbf{A}p)\mathbf{V}\top$ **HS** 5, 12
14. $\mathbf{A}(p \wedge \neg(\neg p)\mathbf{V}\top) \to \neg(\neg\mathbf{A}p)\mathbf{V}\top$ **AND-L** 13, **DIST-ANDA**, **HS**
15. $p \wedge \neg(\neg p)\mathbf{V}\top \to p$ **REFL**, **AND-L**
16. $\mathbf{A}(p \wedge \neg(\neg p)\mathbf{V}\top \to p)$ **R4-A** 15
17. $\mathbf{A}(p \wedge \neg(\neg p)\mathbf{V}\top) \to \mathbf{A}p$ **A13-A**, **R1-MP** 16
18. $\mathbf{A}(p \wedge \neg(\neg p)\mathbf{V}\top) \to \mathbf{A}p \wedge \neg(\neg\mathbf{A}p)\mathbf{V}\top$ **AND-R** 14, 17

19. $\mathbf{AG}p \to \mathbf{GA}p$ — **D9-G** 18

(COM-XA) $\vdash_{\Delta}^{PBTL}$ $\mathbf{AX}p \to \mathbf{XA}p$ (commutativity of $\mathbf{A}$ and $\mathbf{X}$)

Proof:

1. $(\mathbf{E}(\neg p))\mathbf{V}\bot \to \mathbf{E}((\neg p)\mathbf{V}\bot)$ — **A16-EV**
2. $(\neg\mathbf{A}(\neg\neg p))\mathbf{V}\bot \to \neg\mathbf{A}(\neg(\neg p)\mathbf{V}\bot)$ — **D11-E** 1
3. $\mathbf{X}(\neg\mathbf{A}(\neg\neg p)) \to \neg\mathbf{A}(\neg\mathbf{X}(\neg p))$ — **D6-X** 2
4. $\neg\mathbf{X}(\mathbf{A}(\neg\neg p)) \to \mathbf{X}(\neg\mathbf{A}(\neg\neg p))$ — **FUN-X**, **IFF-E**
5. $\neg\mathbf{X}(\mathbf{A}(\neg\neg p)) \to \neg\mathbf{A}(\neg\mathbf{X}(\neg p))$ — **HS** 4, 3
6. $\mathbf{A}(\neg\mathbf{X}(\neg p)) \to \mathbf{X}(\mathbf{A}(\neg\neg p))$ — **A3-N**, **R1-MP** 5
7. $\mathbf{A}(\neg\neg p \to p)$ — **DOUB**, **R4-A**
8. $\mathbf{A}(\neg\neg p \to p) \to (\mathbf{A}(\neg\neg p) \to \mathbf{A}p)$ — **A13-A**
9. $\mathbf{A}(\neg\neg p) \to \mathbf{A}p$ — **R1-MP** 7, 8
10. $\mathbf{G}(\mathbf{A}(\neg\neg p) \to \mathbf{A}p)$ — **R2-G** 9
11. $\mathbf{G}(\mathbf{A}(\neg\neg p) \to \mathbf{A}p) \to (\mathbf{XA}(\neg\neg p) \to \mathbf{XA}p)$ — **RPL-GX**
12. $\mathbf{XA}(\neg\neg p) \to \mathbf{XA}p$ — **R1-MP** 10, 11
13. $\mathbf{A}(\neg\mathbf{X}(\neg p)) \to \mathbf{XA}p$ — **HS** 6, 12
14. $\mathbf{X}(\neg p) \to \neg\mathbf{X}p$ — **FUN-X**, **IFF-E**
15. $(\mathbf{X}(\neg p) \to \neg\mathbf{X}(p)) \to (\neg\neg\mathbf{X}(p) \to \neg\mathbf{X}(\neg p))$ — **CONP**
16. $\neg\neg\mathbf{X}p \to \neg\mathbf{X}(\neg p)$ — **R1-MP** 14, 15
17. $\mathbf{X}p \to \neg\neg\mathbf{X}p$ — **DOUB**, **A3-N**, **R1-MP**
18. $\mathbf{X}p \to \neg\mathbf{X}(\neg p)$ — **HS** 16, 17
19. $\mathbf{A}(\mathbf{X}p \to \neg\mathbf{X}(\neg p))$ — **R4-A** 18
20. $\mathbf{A}(\mathbf{X}p \to \neg\mathbf{X}(\neg p)) \to (\mathbf{AX}p \to \mathbf{A}(\neg\mathbf{X}(\neg p)))$ — **A13-A**
21. $\mathbf{AX}p \to \mathbf{A}(\neg\mathbf{X}(\neg p)))$ — **R1-MP** 19, 20
22. $\mathbf{AX}p \to \mathbf{XA}p$ — **HS** 21, 13

(COM-EF) $\vdash_{\Delta}^{PBTL}$ $\mathbf{FE}p \to \mathbf{EF}p$ (commutativity of $\mathbf{E}$ and $\mathbf{F}$)

Proof:

1. $(\mathbf{E}p)\mathbf{V}\top \to \mathbf{E}(p\mathbf{V}\top)$ — **A16-EV**
2. $(\mathbf{E}p)\mathbf{V}\top \to \mathbf{E}p \lor \mathbf{E}(p\mathbf{V}\top)$ — **OR-R** 1
3. $\mathbf{E}p \to \mathbf{E}p \lor \mathbf{E}(p\mathbf{V}\top)$ — **REFL**, **OR-R**
4. $\mathbf{E}p \lor (\mathbf{E}p)\mathbf{V}\top \to \mathbf{E}p \lor \mathbf{E}(p\mathbf{V}\top)$ — **OR-R** 2, 3
5. $\mathbf{E}p \lor \mathbf{E}(p\mathbf{V}\top) \to \mathbf{E}(p \lor p\mathbf{V}\top)$ — **DIST-ORE**, **IFF-E**
6. $\mathbf{FE}p \to \mathbf{EF}p$ — **HS** 4, 5; **D8-F**

(COM-EX) $\vdash_{\Delta}^{PBTL}$ $\mathbf{XE}p \to \mathbf{EX}p$ (commutativity of $\mathbf{E}$ and $\mathbf{X}$)

Proof:

1. $(\mathbf{E}p)\mathbf{V}\bot \to \mathbf{E}(p\mathbf{V}\bot)$ — **A16-EV**
2. $\mathbf{XE}p \to \mathbf{EX}p$ — **D6-X** 1

(IND-AG) $\vdash_{\Delta}^{PBTL}$ $\mathbf{AG}(p \to \mathbf{X}p) \to (p \to \mathbf{XAG}p)$ (branching induction)

Proof:

1. $\mathbf{G}(p \to \mathbf{X}p) \to (\mathbf{G}p \to \mathbf{GX}p)$ — **MON-G**
2. $\mathbf{GX}p \to \mathbf{XG}p$ — **COM-GX**, **IFF-E**

3. $\mathbf{G}(p \to \mathbf{X}p) \to (\mathbf{G}p \to \mathbf{XG}p)$ **LTRAN**, **R1-MP** 2, **HS** 1
4. $\mathbf{AG}(p \to \mathbf{X}p) \to \mathbf{A}(\mathbf{G}p \to \mathbf{XG}p)$ **R4-A** 3, **MON-A**, **R1-MP**
5. $\mathbf{AG}(p \to \mathbf{X}p) \to (\mathbf{G}p \to \mathbf{XAG}p)$ **A17-AU**, **LTRAN**, **R1-MP**, **HS** 4
6. $\mathbf{AG}(p \to \mathbf{X}p) \to (p \to \mathbf{G}p)$ **A14-A**, **A10-G**, **HS**
7. $\mathbf{G}p \to (\mathbf{AG}(p \to \mathbf{X}p) \to \mathbf{XAG}p)$ **PERM**, **R1-MP** 5
8. $p \to (\mathbf{AG}(p \to \mathbf{X}p) \to \mathbf{XAG}p)$ **LTRAN**, **R1-MP** 7, **HS** 6
9. $\mathbf{AG}(p \to \mathbf{X}p) \to (p \to \mathbf{XAG}p)$ **PERM**, **R1-MP** 8

III.4 Classical First-Order Logic

Postulating the axiomatization of *classical first-order logic* (FOL) discussed in Section 2.6, the following theorems over $\Delta \in \mathsf{obj}\ \mathbf{Sig}^{BFOL}$ are provable:

(ALL-E) $\vdash^{FOL}_{\Delta} \forall x \cdot p[x] \to p$

Because $\mathcal{G}^{FOL}(\Delta)$ is defined by induction, each of its sentences has finite length and so contains only a finite number of variables. Since $|\mathcal{V}^{FOL}| = \aleph_0$, there is $y \in \mathcal{V}^{FOL}$ such that $y \notin Free(p)$ for each $p \in \mathcal{G}^{FOL}(\Delta)$. We proceed as follows:

1. $\forall x \cdot p[x] \to p[x\backslash y]$ **A19-∀**
2. $\forall x \cdot p[x] \to \forall y \cdot p[y]$ **R5-∀** 1
3. $\forall y \cdot p[y] \to p[y\backslash x]$ **A19-∀**
4. $\forall x \cdot p[x] \to p$ **HS** 2, 3; **DEF** $[\cdot]$

(MON-∀) $\vdash^{FOL}_{\Delta} \forall x \cdot (p[x] \to q[x]) \to (\forall x \cdot p[x] \to \forall x \cdot q[x])$
(monotonicity of ∀)

Using the same argument about y as in the proof of **ALL-E**, we obtain:

1. $\forall x \cdot (p[x] \to q[x]) \to (p[x\backslash y] \to q[x\backslash y])$ **A19-∀**, **DEF** $[\cdot]$
2. $\forall x \cdot p[x] \to p[x\backslash y]$ **A19-∀**
3. $(\forall x \cdot p[x] \to p[x\backslash y]) \to$ **RTRAN**
   $((p[x\backslash y] \to q[x\backslash y]) \to (\forall x \cdot p[x] \to q[x\backslash y]))$
4. $(p[x\backslash y] \to q[x\backslash y]) \to (\forall x \cdot p[x] \to q[x\backslash y])$ **R1-MP** 2, 3
5. $(p[x\backslash y] \to q[x\backslash y]) \to \forall x \cdot (\forall x \cdot p[x] \to q[x])$ **R5-∀** 4
6. $\forall x \cdot (\forall x \cdot p[x] \to q[x]) \to (\forall x \cdot p[x] \to \forall x \cdot q[x])$ **A20-∀**
7. $(p[x\backslash y] \to q[x\backslash y]) \to (\forall x \cdot p[x] \to \forall x \cdot q[x])$ **HS** 5, 6
8. $\forall x \cdot (p[x] \to q[x]) \to (\forall x \cdot p[x] \to \forall x \cdot q[x])$ **HS** 1, 7

(DUAL-∀∃) $\vdash^{PBTL}_{\Delta} \forall x \cdot (\neg p) \leftrightarrow \neg\exists x \cdot p$ (duality between ∀ and ∃)

(GEN-∀) $\{p[x]\} \vdash^{FOL}_{\Delta} \forall x \cdot p[x]$

Proof:

1. $p[x]$ **Ass**
2. $p[x] \to (\top \to p[x])$ **A1-I**

3. $\top \to p[x]$ **R1-MP** 1, 2
4. $\top \to \forall x \cdot p[x]$ **R5-**$\forall$ 3
5. $\top$ **REFL**, **D1-**$\top$
6. $\forall x \cdot p[x]$ **R1-MP** 5, 4

(REPL-FOL) $\{x \leftrightarrow y\} \vdash_{\Delta}^{FOL} p[q\backslash x] \leftrightarrow p[q\backslash y]$ (replacement)

We prove this theorem by structural induction on $\mathcal{G}^{FOL}(\Delta)$. The proof is essentially the same as for CPL, with the base case considering atoms instead of proposition symbols and with an inductive step taking care of both open and closed sentences. Open sentences are treated as propositions are in the proof of **REPL-CPL**. Here, we have to examine the case where $p \equiv \forall v \cdot r[v]$. For $v \notin Free(q) \cup Free(x) \cup Free(y)$:

1. $x \leftrightarrow y$ **Ass**
2. $r[v][q\backslash x] \leftrightarrow r[v][q\backslash y]$ **Ind. Hyp.** 1
3. $r[v][q\backslash x] \to r[v][q\backslash y]$ **IFF-E** 2
4. $\forall v \cdot (r[v][q\backslash x] \to r[v][q\backslash y])$ **GEN-**$\forall$ 3
5. $\forall v \cdot (r[v][q\backslash x] \to r[v][q\backslash y]) \to$ $(\forall v \cdot (r[v][q\backslash x]) \to \forall v \cdot (r[v][q\backslash y]))$ **MON-**$\forall$
6. $\forall v \cdot (r[v][q\backslash x]) \to \forall v \cdot (r[v][q\backslash y])$ **R1-MP** 4, 5
7. $r[v][q\backslash y] \to r[v][q\backslash x]$ **IFF-E** 2
8. $\forall v \cdot (r[v][q\backslash y] \to r[v][q\backslash x])$ **GEN-**$\forall$ 7
9. $\forall v \cdot (r[v][q\backslash y] \to r[v][q\backslash x]) \to$ $(\forall v \cdot (r[v][q\backslash y]) \to \forall v \cdot (r[v][q\backslash x]))$ **MON-**$\forall$
10. $\forall v \cdot (r[v][q\backslash y]) \to \forall v \cdot (r[v][q\backslash x])$ **R1-MP** 8, 9
11. $\forall v \cdot (r[v][q\backslash x]) \leftrightarrow \forall v \cdot (r[v][q\backslash y])$ **IFF-RL** 6, 10
12. $(\forall v \cdot r[v])[q\backslash x] \leftrightarrow (\forall v \cdot r[v])[q\backslash y]$ **DEF** $[\cdot]$ 11

To justify the case where $v \in Free(q) \cup Free(x) \cup Free(y)$, first note that if $v \in Free(x)$, v would be bound in the substitution and then $v \notin Free(p[q\backslash x])$, which violates the requirement of performing only "free for" substitutions. The same rationale applies to y. When $v \in Free(q)$, the definition of $[\cdot]$ allows us to conclude that the proof above is still acceptable. All these cases entail that the theorem holds for any $\{p, q, x, y\} \subseteq \mathcal{G}^{FOL}(\Delta)$ (such that substitutions can be made).

(MON-$\forall\exists$**)** $\vdash_{\Delta}^{PBTL} \forall x \cdot (p[x] \to q[x]) \to (\exists x \cdot p[x] \to \exists x \cdot q[x])$

Substitute **G** for $\forall$ and **F** for $\exists$ in the proof of **MON-GF** presented in Section I.2 and change the labels that justify each proof step accordingly to obtain a proof of this theorem.

(EXC-$\forall\exists$**)** $\vdash_{\Delta}^{FOL} \forall x \cdot (p[x] \to q) \leftrightarrow (\exists x \cdot p[x] \to q)$ provided that $x \notin Free(q)$.
[or $\exists x \cdot (p[x] \to q) \leftrightarrow (\forall x \cdot p[x] \to q)$]

Proof:

1. $\forall x \cdot (p[x] \to q) \to (p[y] \to q)$ **A19-∀**
2. $\forall x \cdot (p[x] \to q) \to \forall x \cdot (\neg q \to \neg p[x])$ **CONP**, **HS** 1, **R5-∀**
3. $\forall x \cdot (p[x] \to q) \to (\neg q \to \forall x \cdot (\neg p[x]))$ **A20-∀**, **HS** 2
4. $\forall x \cdot (p[x] \to q) \to (\neg\forall x \cdot (\neg p[x]) \to \neg\neg q)$ **CONP**, **HS** 3
5. $(\neg\forall x \cdot (\neg p[x]) \to \neg\neg q) \to (\neg\forall x \cdot (\neg p[x]) \to q)$ **DOUB**, **LTRAN**, **R1-MP**
6. $\forall x \cdot (p[x] \to q) \to (\exists x \cdot p[x] \to q)$ **HS** 4, 5; **D12-∃**
7. $(\exists x \cdot p[x] \to q) \to (\exists x \cdot p[x] \to \neg\neg q)$ **CONP**, **LTRAN**, **R1-MP**
8. $(\exists x \cdot p[x] \to q) \to (\neg q \to \forall x \cdot (\neg p[x]))$ **D12-∃** 7, **A3-N**, **R1-MP**
9. $(\neg q \to \forall x \cdot (\neg p[x])) \to (\neg q \to \neg p[y])$ **A19-∀**, **LTRAN**, **R1-MP**
10. $(\neg q \to \forall x \cdot (\neg p[x])) \to \forall x \cdot (p[x] \to q)$ **A3-N**, **HS** 9, **R5-∀**
11. $(\exists x \cdot p[x] \to q) \to \forall x \cdot (p[x] \to q)$ **HS** 8, 10
12. $(\exists x \cdot p[x] \to q) \leftrightarrow (\exists x \cdot p[x] \to \neg\neg q)$ **IFF-RL** 6, 11

(MOV-IF∀) $\vdash^{FOL}_{\Delta}$ $\forall x \cdot (p \to q[x]) \leftrightarrow (p \to \forall x \cdot q[x])$ provided that $x \notin Free(p)$.
(MOV-IF∃) [or $\exists x \cdot (p \to q[x]) \leftrightarrow (p \to \exists x \cdot q[x])$]

(DIST-AND∀) $\vdash^{FOL}_{\Delta}$ $\forall x \cdot (p[x] \wedge q[x]) \leftrightarrow \forall x \cdot p[x] \wedge \forall x \cdot q[x]$
(distribution of ∀ over ∧)

Proof:

1. $\forall x \cdot p[x] \wedge q[x] \to p[x] \wedge q[x]$ **ALL-E**
2. $p[x] \to p[x]$ **REFL**
3. $p[x] \wedge q[x] \to p[x]$ **AND-L** 2
4. $\forall x \cdot p[x] \wedge q[x] \to p[x]$ **HS** 1, 3
5. $\forall x \cdot (\forall x \cdot p[x] \wedge q[x] \to p[x])$ **GEN-∀** 4
6. $\forall x \cdot (\forall x \cdot p[x] \wedge q[x] \to p[x]) \to \forall x \cdot p[x] \wedge q[x] \to \forall x \cdot p[x]$ **MON-∀**
7. $\forall x \cdot p[x] \wedge q[x] \to \forall x \cdot p[x]$ **R1-MP** 5, 6
8. $q[x] \to q[x]$ **REFL**
9. $p[x] \wedge q[x] \to q[x]$ **AND-L** 8
10. $\forall x \cdot p[x] \wedge q[x] \to q[x]$ **HS** 7, 9
11. $\forall x \cdot (\forall x \cdot p[x] \wedge q[x] \to q[x])$ **GEN-∀** 10
12. $\forall x \cdot (\forall x \cdot p[x] \wedge q[x] \to q[x]) \to \forall x \cdot p[x] \wedge q[x] \to \forall x \cdot q[x]$ **MON-∀**
13. $\forall x \cdot p[x] \wedge q[x] \to \forall x \cdot q[x]$ **R1-MP** 11, 12
14. $\forall x \cdot p[x] \wedge q[x] \to \forall x \cdot p[x] \wedge \forall x \cdot q[x]$ **AND-R** 7, 13
15. $\forall x \cdot p[x] \to \forall x \cdot p[x]$ **REFL**
16. $\forall x \cdot p[x] \wedge \forall x \cdot q[x] \to \forall x \cdot p[x]$ **AND-L** 15
17. $\forall x \cdot p[x] \to p[x]$ **ALL-E**
18. $\forall x \cdot p[x] \wedge \forall x \cdot q[x] \to p[x]$ **HS** 16, 17
19. $\forall x \cdot q[x] \to \forall x \cdot q[x]$ **REFL**
20. $\forall x \cdot p[x] \wedge \forall x \cdot q[x] \to \forall x \cdot q[x]$ **AND-L** 19
21. $\forall x \cdot q[x] \to q[x]$ **ALL-E**
22. $\forall x \cdot p[x] \wedge \forall x \cdot q[x] \to q[x]$ **HS** 20, 21
23. $\forall x \cdot p[x] \wedge \forall x \cdot q[x] \to p[x] \wedge q[x]$ **AND-R** 18, 22
24. $\forall x \cdot (\forall x \cdot p[x] \wedge \forall x \cdot q[x] \to p[x] \wedge q[x])$ **GEN-∀** 23
25. $\forall x \cdot p[x] \wedge \forall x \cdot q[x] \to \forall x \cdot p[x] \wedge q[x]$ **MON-∀**, **R1-MP** 24
26. $\forall x \cdot (p[x] \wedge q[x]) \leftrightarrow \forall x \cdot p[x] \wedge \forall x \cdot q[x]$ **IFF-RL** 14, 25

(DIST-OR∃) $\vdash^{FOL}_{\Delta} \exists x \cdot (p[x] \vee q[x]) \leftrightarrow \exists x \cdot p[x] \vee \exists x \cdot q[x]$
(distribution of ∃ over ∨)

Proof:

1. $\forall x \cdot (\neg p[x] \wedge \neg q[x]) \leftrightarrow \forall x \cdot \neg p[x] \wedge \forall x \cdot \neg q[x]$ **DIST-AND∀**
2. $\forall x \cdot (\neg p[x] \wedge \neg q[x]) \to \forall x \cdot \neg p[x] \wedge \forall x \cdot \neg q[x]$ **IFF-E** 1
3. $(\forall x \cdot (\neg p[x] \wedge \neg q[x]) \to \forall x \cdot \neg p[x] \wedge \forall x \cdot \neg q[x]) \to$ $(\neg(\forall x \cdot \neg p[x] \wedge \forall x \cdot \neg q[x]) \to \neg\forall x \cdot (\neg p[x] \wedge \neg q[x]))$ **CONP**
4. $\neg(\forall x \cdot \neg p[x] \wedge \forall x \cdot \neg q[x]) \to \neg\forall x \cdot (\neg p[x] \wedge \neg q[x])$ **R1-MP** 2, 3
5. $\neg\forall x \cdot \neg p[x] \vee \neg\forall x \cdot \neg q[x] \leftrightarrow \neg(\forall x \cdot \neg p[x] \wedge \forall x \cdot \neg q[x])$ **DM**
6. $\neg\forall x \cdot \neg p[x] \vee \neg\forall x \cdot \neg q[x] \to \neg(\forall x \cdot \neg p[x] \wedge \forall x \cdot \neg q[x])$ **IFF-E** 5
7. $\neg\forall x \cdot \neg p[x] \vee \neg\forall x \cdot \neg q[x] \to \neg\forall x \cdot (\neg p[x] \wedge \neg q[x])$ **HS** 6, 4
8. $\neg(p[x] \vee q[x]) \leftrightarrow \neg p[x] \wedge \neg q[x]$ **DM**
9. $\neg\forall x \cdot (\neg p[x] \wedge \neg q[x]) \to \neg\forall x \cdot \neg(p[x] \vee q[x])$ **REPL-FOL** 8, **IFF-E**
10. $\exists x \cdot p[x] \vee \exists x \cdot q[x] \to \exists x \cdot (p[x] \vee q[x])$ **HS** 7, 9; **D12-∃**
11. $\forall x \cdot \neg p[x] \wedge \forall x \cdot \neg q[x] \to \forall x \cdot (\neg p[x] \wedge \neg q[x])$ **IFF-E** 1
12. $(\forall x \cdot \neg p[x] \wedge \forall x \cdot \neg q[x] \to \forall x \cdot (\neg p[x] \wedge \neg q[x])) \to$ $(\neg\forall x \cdot (\neg p[x] \wedge \neg q[x]) \to \neg(\forall x \cdot \neg p[x] \wedge \forall x \cdot \neg q[x]))$ **CONP**
13. $\neg\forall x \cdot (\neg p[x] \wedge \neg q[x]) \to \neg(\forall x \cdot \neg p[x] \wedge \forall x \cdot \neg q[x])$ **R1-MP** 11, 12
14. $\neg\forall x \cdot \neg(p[x] \vee q[x]) \to \neg\forall x \cdot (\neg p[x] \wedge \neg q[x])$ **REPL-FOL** 8, **IFF-E**
15. $\neg\forall x \cdot \neg(p[x] \vee q[x]) \to \neg(\forall x \cdot \neg p[x] \wedge \forall x \cdot \neg q[x])$ **HS** 14, 13
16. $\neg(\forall x \cdot \neg p[x] \wedge \forall x \cdot \neg q[x]) \to \neg\forall x \cdot \neg p[x] \vee \neg\forall x \cdot \neg q[x]$ **IFF-E** 5
17. $\exists x \cdot (p[x] \vee q[x]) \to \exists x \cdot p[x] \vee \exists x \cdot q[x]$ **HS** 15, 16; **D12-∃**
18. $\exists x \cdot (p[x] \vee q[x]) \leftrightarrow \exists x \cdot p[x] \vee \exists x \cdot q[x]$ **IFF-RL** 10, 17

III.5 Many-Sorted Logic with Equality

Postulating the axiomatization of *many-sorted logic with equality* ($MSFOL$) discussed in Section 2.6.1, the following theorems over $\Delta \in \mathsf{obj}\ \mathbf{Sig}^{MSFOL}$ are provable:

(REPL-FOL) $\{x \leftrightarrow y\} \vdash^{MSFOL}_{\Delta} p[q\backslash x] \leftrightarrow p[q\backslash y]$ (replacement)

The proof of this theorem is developed by structural induction on $\mathcal{G}^{FOL}(\Delta)$ as usual and is precisely the same as for FOL, with the base case considering equality of terms as another type of atom. Details are omitted here.

(REFL-EQ) $\vdash^{MSFOL}_{\Delta} x = y \to y = x$ (reflexivity of equality)

Proof:

1. $x = y \to (x = x \to y = x)$ **A22-EQ**
2. $(x = y \to (x = x \to y = x)) \to (x = x \to (x = y \to y = x))$ **PERM**
3. $x = x \to (x = y \to y = x)$ **R1-MP** 1, 2
4. $x = x$ **A21-EQ**
5. $x = y \to y = x$ **R1-MP** 4, 3

(TRAN-EQ) $\vdash^{MSFOL}_{\Delta} x = y \wedge y = z \to x = z$ (transitivity of equality)

Proof:

1. $x = y \to (x = z \to y = z)$ **A22-EQ**
2. $x = y \wedge x = z \to (x = z \to y = z)$ **AND-L** 1
3. $(x = y \wedge x = z \to (x = z \to y = z)) \to$ **A2-I**
   $((x = y \wedge x = z \to x = z) \to (x = y \wedge x = z \to y = z))$
4. $(x = y \wedge x = z \to x = z) \to (x = y \wedge x = z \to y = z)$ **R1-MP** 2, 3
5. $x = z \to x = z$ **REFL**
6. $x = y \wedge x = z \to x = z$ **AND-L** 5
7. $x = y \wedge x = z \to y = z$ **R1-MP** 6, 4

III.6 First-Order Temporal Logic

Postulating the axiomatization of *linear time many-sorted first-order logic with equality* ($LTMSL$) discussed in Section 2.7, the following theorems over $\Delta \in$ obj $\mathbf{Sig}^{LTMSL}$ are provable:

(FUN) $\vdash_{\Delta}^{LTMSL} f(x_1, \ldots, x_n) = x \wedge f(x_1, \ldots, x_n) = y \to x = y$
for any $f \in Funct(\Delta) \cup Attr(\Delta)$ with $arity(f) = n$.

This is a direct consequence of **TRAN-EQ**.

(BARC-G) $\vdash_{\Delta}^{LTMSL} \forall x \cdot \mathbf{G}(p[x]) \leftrightarrow \mathbf{G}(\forall x \cdot p[x])$ (Barcan for **G**)

Substitute $\mathbf{A}p$ for $\forall x \cdot p[x]$ in the proof of **COM-AG** presented in Section I.3 and change the labels that justify each proof step accordingly to obtain a proof of this theorem in one direction. The converse is obtained from the properties of $\forall$ and **G**.

(BARC-X) $\vdash_{\Delta}^{LTMSL} \forall x \cdot \mathbf{X}(p[x]) \leftrightarrow \mathbf{X}(\forall x \cdot p[x])$ (Barcan for **X**)
[or $\exists x \cdot \mathbf{X}(p[x]) \leftrightarrow \mathbf{X}(\exists x \cdot p[x])$]

Substitute $\mathbf{A}p$ for $\forall x \cdot p[x]$ in the proof of **COM-AX** presented in Section I.3 and change the labels that justify each proof step accordingly to obtain a proof of the direct case of the first form of this theorem. The converse in this case is obtained from the properties of $\forall$ and **X**. The second form is directly obtained from the first one due to the functionality of **X** (**FUN-X**).

(BARC-F) $\vdash_{\Delta}^{LTMSL} \mathbf{F}(\exists x \cdot p[x]) \leftrightarrow \exists x \cdot \mathbf{F}(p[x])$ (Barcan for **F**)

Substitute $\mathbf{E}p$ for $\exists x \cdot p[x]$ in the proof of **COM-EF** presented in Section I.3 and change the labels that justify each proof step accordingly to obtain a proof of this theorem. The converse is obtained from the properties of $\exists$ and **F**.

(BARC-GF) $\vdash_{\Delta}^{LTMSL}$ $\mathbf{GF}(\exists x \cdot p) \leftrightarrow \exists x \cdot \mathbf{GF}p$ (Barcan for **GF**)

Proof:

1.	$\forall x \cdot \mathbf{G}p \to \forall x \cdot \mathbf{G}p$	**REFL**
2.	$\mathbf{G}p \to \mathbf{FG}p$	**REFL**, **OR-L**, **D7-F**
3.	$\forall x \cdot \mathbf{G}p \to \forall x \cdot \mathbf{FG}p$	**GEN-∀** 2, **MON-∀**, **R1-MP**, **HS** 1
4.	$\mathbf{F}(\forall x \cdot \mathbf{G}p) \to \mathbf{F}(\forall x \cdot \mathbf{FG}p)$	**R2-G** 3, **MON-GF**, **R1-MP**
5.	$\mathbf{F}(\forall x \cdot \mathbf{FG}p) \to \mathbf{FFG}p$	**ALL-E**, **R2-G**, **MON-GF**, **R1-MP**
6.	$\mathbf{F}(\forall x \cdot \mathbf{FG}p) \to \forall x \cdot \mathbf{FG}p$	**IDEM-F**, **HS** 5, **R5-∀**
7.	$\mathbf{F}(\forall x \cdot \mathbf{G}p) \to \forall x \cdot \mathbf{FG}p$	**HS** 4, 6
8.	$\exists x \cdot \mathbf{GF}p \to \mathbf{G}(\exists x \cdot \mathbf{F}p)$	**INVE**, **R1-MP** 7, **DUAL-∀∃**, **DUAL-GF**
9.	$\mathbf{G}(\exists x \cdot \mathbf{F}p) \to \mathbf{GF}(\exists x \cdot p)$	**BARC-F**, **R2-G**, **MON-G**, **R1-MP**
10.	$\exists x \cdot \mathbf{GF}p \to \mathbf{GF}(\exists x \cdot p)$	**HS** 8, 9
11.	$\forall x, y \cdot \mathbf{G}(\mathbf{G}(\neg p[x]) \to \neg p[y]) \vee$ $\mathbf{G}(\mathbf{G}(\neg p[y]) \to \neg p[x])$	**LIN-G**, **GEN-∀**
12.	$\forall x, y \cdot \mathbf{G}(\mathbf{G}(\neg p[x]) \to \neg p[y])$	**GEN-∀**, **MON-∀**, **R1-MP** 11
13.	$\forall x, y \cdot \mathbf{G}(\neg \mathbf{F}(p[x]) \to \neg p[y])$	**DUAL-GF**, **R1-MP** 12
14.	$\forall x, y \cdot \mathbf{G}(p[y] \to \mathbf{F}(p[x]))$	**INVE**, **R1-MP** 13
15.	$\forall x \cdot \mathbf{G}(\forall x \cdot p[y] \to \mathbf{F}(p[x]))$	**BARC-G**, **R1-MP** 14
16.	$\forall x \cdot \mathbf{G}(\exists x \cdot p[y] \to \mathbf{F}(p[x]))$	**EXC-∀∃**, **R1-MP** 15
17.	$\forall x \cdot \mathbf{GG}(\exists x \cdot p[y] \to \mathbf{F}(p[x]))$	**IDEM-G**, **R1-MP** 16
18.	$\forall x \cdot \mathbf{G}(\mathbf{F}(\exists x \cdot p[y]) \to \mathbf{F}(p[x]))$	**MON-GF**, **IDEM-F**, **R1-MP** 17
19.	$\forall x \cdot \mathbf{GF}(\exists x \cdot p[y]) \to \mathbf{GF}(p[x])$	**MON-G**, **R1-MP** 18
20.	$\mathbf{GF}(\exists x \cdot p[y]) \to \exists x \cdot \mathbf{GF}(p[x])$	**MON-∀∃**, **R1-MP** 19

(BARC-FG) $\vdash_{\Delta}^{LTMSL}$ $\forall x \cdot \mathbf{FG}p \leftrightarrow \mathbf{FG}(\forall x \cdot p)$ (Barcan for **FG**)

(BARC-A) $\vdash_{\Delta}^{LTMSL}$ $\forall x \cdot \mathbf{A}p \leftrightarrow \mathbf{A}(\forall x \cdot p)$ (Barcan for **A**)

(BARC-E) $\vdash_{\Delta}^{LTMSL}$ $\mathbf{E}(\exists x \cdot p) \leftrightarrow \exists x \cdot \mathbf{E}p$ (Barcan for **E**)